Orbit Mechanics I

E.J.O. Schrama
Delft University of Technology,
Faculty of aerospace engineering, Astrodynamics and Satellite systems
e-mail: e.j.o.schrama@lr.tudelft.nl

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1 Introduction

These notes belong to class ae4-e01 which is on satellite observation systems and reference systems. The lectures are in the 4th year Master of Science course on Earth and Planetary observation at the Faculty of Aerospace Engineering at the Delft University of Technology in the Netherlands.

1.1 Our solar system

The first astronomic observations were made more than two millenia ago, the quality of observation of the sky was constrained to the optical resolution and the sensitivity of the human eye. The brightness of a star is usually indicated by its magnitude, a change in the magnitude of 1 corresponds to a change in brightness of 2.5. Under ideal conditions the human eye is limited to magnitude six, and the optical resolution is roughly 15" (thus 15 seconds of arc), while the optical resolution of binoculars is 2.5". The naked eye is already a very sensitive and high quality optical instrument for basic astronomic observations. We are able to distinguish planets from Mercury to Saturn, comets, meteors and satellites but our naked-eye lacks the optical resolution to observe the 4 largest moons of Jupiter.

The discussion on the motion of planets along the night sky goes back to ancient history. The Greeks and Romans associated the planets with various gods. Mars was for instance the God of War, Jupiter held the same role as Zeus in the Greek Pantheon and Mercury was the God of trade, profit and commerce. Planets are unique in the night sky since the wander relative to the stars, who seem to be fixed on a celestial sphere for an observer on a non-rotating Earth. As soon as you observe the real thing from a rotating Earth everything become even more confusing, which introduced the model that the Earth was the center of the universe, that it was flat and that you could fall over the horizon and that everything else in the universe was rotating around it. The believe has held-up until the invention of the telescope in 1608 and its first application for astronomic observations in 1610 by Galileo Galilei.

1.2 Galileo, Copernicus, Brahe and Kepler

Galileo Galilei was an Italian astronomer who lived between 1564 and 1642 who was renowned for his revolutionary new concept the solar system causing him to get into trouble with the inquisition. He modified the then existing telescope into an instrument suitable for astronomic observations to conclude in 1610 that there are four Moons orbiting the planet Jupiter. The telescope was earlier invented by the German-born Dutch eyeglass maker Hans Lippershey who demonstrated the concept of two refracting lenses to the Dutch parliament in 1608. After all it is not surprising that the observation of moons around Jupiter was made in southern Europe, which on the average has a higher chance of clear night skies compared to the Netherlands. One of Galileo Galilei's comments on the classical view on the solar system was that his instrument permitted him to see moons orbiting another planet, and that the classical model was wrong.

Other developments took place around the same time in Europe. Nicolaus Copernicus was a Polish astronomer who lived from 1473 to 1543 and he formulated the concept of planets wandering in circular orbits about the Sun, which was new compared to the traditional geocentric models of Claudius Ptolomaeus (who lived from 87 to 150) and the earlier model of Hypparchus (190 to 120 BC). It was the Danish astronomer Tycho Brahe who lived from 1546 to 1601 to conclude on basis of observations of the planet Mars that there were deviations from the Copernican model of the solar system. The observations of Tycho Brahe assisted the German mathematician, astronomer and astrologer Johannes Kepler who lived from 1571 to 1630 to complete a more fundamental model the explains the motion of planets in our solar system. The Keplerian model is still used today because it is sufficiently accurate to provide short-term and first-order descriptions of planetary ephemerides in our solar system and satellites orbiting the Earth.

1.3 Kepler's laws

The mathematical and physical model of the solar system is summarized in three laws postulated by Kepler. The first and the second law were published in Astronomia Nova in 1609, while the third law was published in Harmonices Mundi in 1619:

- Law I: In our solar system, the Sun is in a focal point of an ellipse, and the planets move in an orbital plane along this ellipse, see plate 1.
- Law II: The ratio of an area swept by a planet relative to the time required is a constant, see plate 2.
- Law III: The square of the mean orbital motion times the cube of the largest circle containing the ellipse is constant. Thus:

  $$n^2 a^3 = G.M = \mu \tag{1}$$

  where n is the mean motion in radians per second and a the semi-major axis in some unit of length. In this equation G is the universal gravitational constant and M is the mass of the Sun. (both in units that correspond to the left hand side).

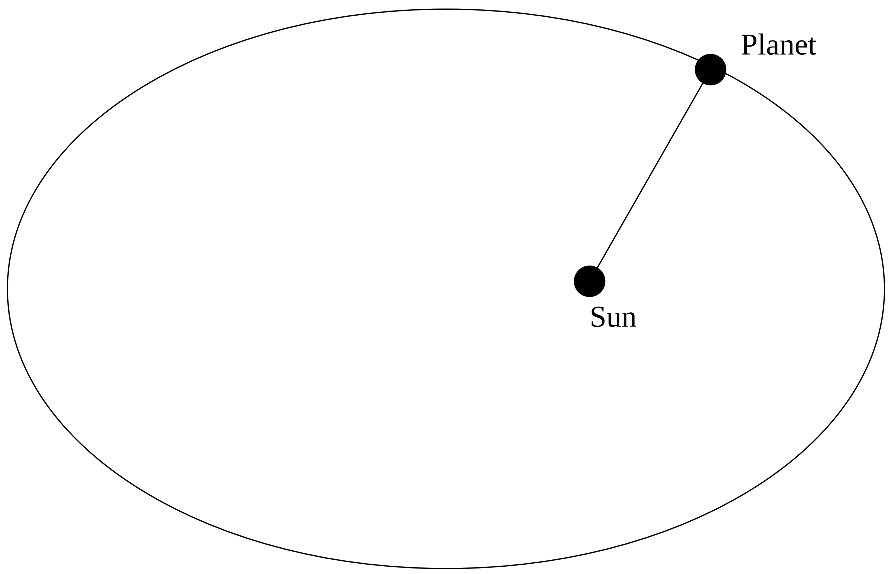

Figure 1: Elliptical orbit of a planet around the sun in one of the focal points

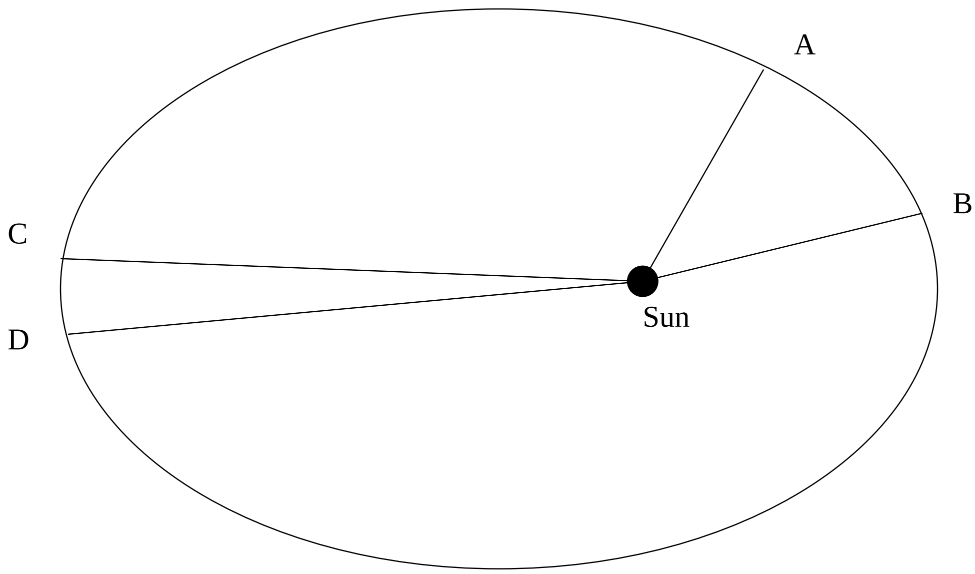

Figure 2: Kepler's equal area law: segment AB-Sun and segment CD-Sun span equal areas, the motion of the planet between A and B takes as long as it would between C and D

2 The Keplerian model

In this section we demonstrate the validity of the Keplerian model, essentially by returning to the equations of motion of a satellite and by substitution of a suitable gradient of a potential function in these equations. This will result in an expression that describes the radius of the planet that depends on its position in orbit. After this point we will derive a similar expression for the scalar velocity in relation to the radius, the latter is called the vis-viva equation.

2.1 Equations of motion

In an inertial coordinate system the generalized equation of motion of a satellite is:

$$\ddot{\overline{x}} = -\nabla V + \sum_i \overline{f}^i \tag{2}$$

where $\ddot{\overline{x}}$ is an acceleration vector and V a potential function and where the terms $\overline{f}^i$ represent small additional accelerations. Equation (2) is a second-order ordinary differential equations explaining that a particle in a force field is accelerating along the local direction of gravity (which is the gradient of V in the model). The model allows for additional accelerations which are usually much smaller than the gravity effect.

A falling object on Earth like a bullet leaving a gun barrel will exactly obey these equations. In this case gravity is the main force that determines the motion, while in addition air drag plays a significant role. One way to obtain a satellite in orbit would be to shoot the bullet with sufficient horizontal velocity over the horizon. If there wouldn't be air drag then Kepler's orbit model predicts that this particular bullet eventually hits the gunman in his back.

There are at least two reasons why this will never happen. The first reason is of course the presence of air drag, the second reason is that the coordinate frame we live in experiences a diurnal motion caused by a rotation Earth. (It is up to you to verify that "Kepler's bullet" will hit an innocent bystander roughly 2000 km west of your current location on the equator.) Air drag will keep the average bullet exiting a barrel within a range of about 2 kilometer which is easy to verify if you implement eq. (2) as a system of first order ordinary differential equations in MATLAB. The rotating Earth causes a much smaller effect and you will not easily notice it. (In reality cross-wind has a more significant effect).

In order to demonstrate the consequences of rotating Earth you are better off with the pendulum of Foucault. Jean Bernard Léon Foucault was a French physicist who lived from 1819 to 1868 and he demonstrated the effect of Earth rotation on a pendulum mounted in the Meridian Room of the Paris observatory in 1851, today the pendulum can be found in the Panthéon in Paris where it is a 28-kg metal ball suspended by wire in the dome of this building. Foucault's pendulum will oscillate in an orbital plane, due to the Coriolis forces that act on the pendulum we observe a steady shift of this orbital plane that depends on the latitude of the pendulum.

Some important fact are:

- The coordinate system referred to in equation (2) is an inertial coordinate system that does not allow frame accelerations due to frame motion or rotation.

- Whenever we speak about gravity on the Earth's surface, as we all know it, we refer to the sum of gravitational and rotational acceleration.
- The potential V in equation (2) is thus best referred to as a gravitational potential, sometimes it is also called the geopotential.

The concept of potential functions is best explained in a separate lecture on potential theory. The following section described the basic properties to arrive at a suitable potential function for the Kepler problem.

2.2 Potential functions

In this case we speak about a scalar function that describes the potential energy that is independent of the mass of a small object moving around in a gravitational force field. The energy that we refer is related to the problem of moving around in a force field, whereby the field itself is caused by the gravitational attraction of a mass source that is far larger than the small object moving around. The potential at the end of the path minus the potential at the beginning of the path is equal to the amount of Joules per kg that we need to put in the motion that takes place in this force field. If you move away from the source mass you have to push the object, which costs energy. But instead, when you approach the source mass then all this energy comes back again for free, and if you move along surfaces of equal potential energy then no extra energy is needed to move around. Force fields that possess this properties are said to be conservative.

Mathematically speaking this means that the Laplacian of the potential V is zero, and thus that $\nabla^2 V = 0$. To explain why this is the case we go back to the Gauss integral theorem which states:

$$\int_{\Omega} (\nabla, \vec{w})\, d\sigma = \oint_{\Omega'} (\vec{w}, \vec{n})\, d\sigma' \tag{3}$$

Here Ω is an arbitrary volume in space and Ω' is a surface element on this volume. Furthermore $\vec{n}$ is an vector of length 1 that is directed outwards on a surface element, while $\vec{w}$ is an arbitrary vector function. If we take $\vec{w}$ as the gradient of the potential V, and if we stay outside the masses that generate V then:

$$\int_{\Omega} (\nabla, \nabla V)\, d\sigma = \oint_{\Omega'} (\nabla V, \vec{n})\, d\sigma' \tag{4}$$

In a conservative force field the right hand side of this integral relation will vanish for any arbitrary volume Ω that does not overlap with the masses V. If we take an infinitesimal small volume Ω then the left hand side becomes:

$$\nabla^2 V = \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{5}$$

This equation is known as the Laplace equation, potential functions V that fulfill the Laplace equation are said to generate a conservative force field ∇V. And within such a conservative force field you can always wander around without losing any energy. Non-conservative force fields also exist, in this case you would lose energy by moving around a closed path within the field.

In physics all electric, magnetic and gravitational field are conservative. Gravitation is unique in the sense that it doesn't interact with electric and magnetic fields. The latter two fields do interact, the most general interaction between E and B is described by the Maxwell equations that permit Electro-Magnetic waves. Gravitation does not permit waves, at least, not in Newtonian physics. The theory of general relativity does allow for gravity waves, although these waves have not yet been detected. Other effects caused by general relativity such as the peri-helium precession of the planet Mercury or the gravitational bending of light have been demonstrated.

The concept "gravity wave" is also used in non-relativistic physics, and for instance in the solution of the Navier Stokes equations. In this case we call a surface wave in a hydrodynamic model a gravity wave because gravity is the restoring force in the dynamics.

2.3 Solutions of the Laplace equation

A straightforward solution of V that fulfills the Laplace equation is the function $V = -\mu/r$ where r is the radius from any arbitrary point in space relative to a source point mass. Later we will show that this point mass potential function applies to the Kepler problem.

The minus sign in front of the gradient operator in equation 2 depends on the convention that we use for defining the geopotential V. If we would start at the Earth's surface the potential would attain a value V_a, and at some height above the surface it would be V_b. The difference between $V_b - V_a$ should in this case be positive, because we had to put a certain number of Joules per kilogram to get it from a to b, and this can only be the case is V_b is greater than V_a. Once we have traveled from the Earth's surface to infinity there is no more energy required to move around. Thus we must demand that $V = 0$ at infinite distance from the source mass.

The $V = -\mu/r$ potential function is one of the many possible solutions of the Laplace equation. We call it the point mass potential function. There are higher order moments of the potential function. In this case we use series of spherical harmonics which are base functions consisting of Legendre polynomial multiplied times geometric functions. For the moment this problem is deferred until we need to refine variations in the gravitational field that differ from the central force field.

2.4 Keplerian equations of motion

A suitable potential V for the Kepler model is:

$$V(r) = -\frac{\mu}{r} \tag{6}$$

It is up to the reader to confirm that this function fulfills the Laplace equation, but also, that it attains a value of zero at $r = \infty$ where r is the distance to the point mass and where $\mu = G.M$ with G is representing the universal gravitational constant and M the mass which are both positive constants.

The gradient of V is the gravitational acceleration vector that we will substitute in the general equations of motion (2), which in turn explains that a satellite or planet at (x, y, z) will experience an acceleration $(\ddot{x}, \ddot{y}, \ddot{z})$ which agrees with the direction indicated by the negative gradient $-\nabla$ of the potential function $V = -\mu/r$. The equations of motion in (2)

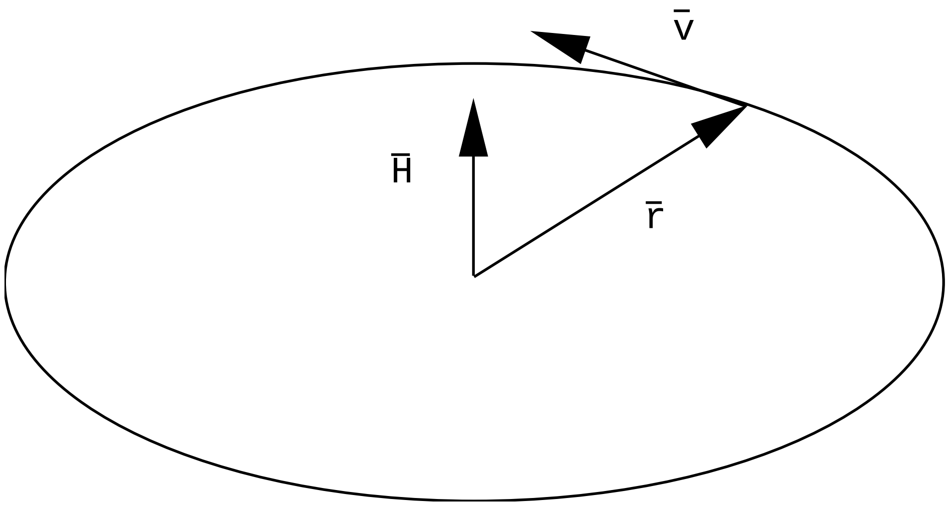

Figure 3: The angular momentum vector is obtained by the vector cross product of the position and velocity vector.

may now be rearranged as:

$$\begin{aligned} \ddot{x} &= \frac{\partial V}{\partial x} + \sum_i f_x^i \\ \ddot{y} &= \frac{\partial V}{\partial y} + \sum_i f_y^i \\ \ddot{z} &= \frac{\partial V}{\partial z} + \sum_i f_z^i \end{aligned} \tag{7}$$

which becomes:

$$\begin{array}{rclcrcl} \partial\dot{x}/\partial t &=& -\mu x/r^3 & \quad & \partial x/\partial t &=& \dot{x} \\ \partial\dot{y}/\partial t &=& -\mu y/r^3 & & \partial y/\partial t &=& \dot{y} \\ \partial\dot{z}/\partial t &=& -\mu z/r^3 & & \partial z/\partial t &=& \dot{z} \end{array} \tag{8}$$

2.5 Orbit plane

So far we have assumed that x y and z are inertial coordinates, and that the motion of the satellite or planet takes place in a three dimensional space. The remarkable observation of Kepler was that the motion occurs within in a plane that intersects the center of the point source mass generating V. This plane is called the orbit plane, and the interested reader may ask why this is the case. To understand this problem we need to consider the angular momentum vector $\overline{H}$ which is obtained as:

$$\overline{r} \times \overline{v} = \overline{x} \times \dot{\overline{x}} = \overline{H} \tag{9}$$

where $\overline{v}$ is the velocity vector and $\overline{r}$ the position vector, see also figure 3. If we assume that $\overline{x} = \overline{r} = (x, y, 0)$ and that $\dot{\overline{x}} = \overline{v} = (\dot{x}, \dot{y}, 0)$ then:

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \times \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x\dot{y} - y\dot{x} \end{bmatrix}$$

which explains that the angular momentum vector is perpendicular to the plane spanned by $\overline{r}$ and $\overline{v}$. To demonstrate that $\dot{\overline{H}} = 0$ we evaluate:

$$\frac{\partial}{\partial t}\left(\dot{\overline{x}} \times \overline{x}\right) = \left(\ddot{\overline{x}} \times \overline{x}\right) + \left(\dot{\overline{x}} \times \dot{\overline{x}}\right)$$

The last term is zero, due to the fact that:

$$\ddot{\overline{x}} = -\frac{\mu}{r^3}\overline{x}$$

we also find that:

$$\ddot{\overline{x}} \times \overline{x} = 0$$

so that $\dot{\overline{H}} = 0$. A direct consequence is that we conserve angular momentum, and thus momentum and energy. This proof also explains why Kepler found an equal area law and why the motion is confined to an orbital plane.

2.6 Substitution 1

To simplify the search for a solution we confine ourself to an orbital place. A convenient choice is in this case to work in polar coordinates so that:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

In the sequel we will substitute this expression in the equations of motion that follow from the point mass potential, see also equation (8). An intermediate step is:

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}$$

so that:

$$\begin{aligned} \ddot{x} &= \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ \ddot{y} &= \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \end{aligned}$$

which is equivalent to:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{bmatrix} \tag{10}$$

For the gradient we have:

$$\begin{bmatrix} \partial V/\partial x \\ \partial V/\partial y \end{bmatrix} = \begin{bmatrix} \partial r/\partial x & \partial\theta/\partial x \\ \partial r/\partial y & \partial\theta/\partial y \end{bmatrix} \begin{bmatrix} \partial V/\partial r \\ \partial V/\partial\theta \end{bmatrix} \tag{11}$$

so that:

$$\begin{bmatrix} \partial V/\partial x \\ \partial V/\partial y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} -\mu/r^2 \\ 0 \end{bmatrix} \tag{12}$$

Since the right hand sides of (11) and (12) are equal we get:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \tag{13}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{14}$$

For the length of the angular momentum vector we get:

$$\begin{aligned} h = |\overline{H}| &= x\dot{y} - y\dot{x} \\ &= +r\cos\theta(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) - r\sin\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) \\ &= r^2\dot{\theta} \end{aligned}$$

which demonstrates again that equal areas are covered in equal units of time in Kepler's second law. Since h is constant we obtain after differentiation with respect to time:

$$\dot{h} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0 \tag{15}$$

Since $r = 0$ is a trivial solution we keep:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{16}$$

which is equal to (14). This consideration does not lead to a new insight in the problem. And thus we turn our attention to eq. (13) which we can solve with a new substitution of parameters.

2.7 Substitution 2

At this point a suitable parameter substitution is $r = 1/u$ and some convenient partial derivatives are:

$$\frac{\partial u}{\partial r} = -\frac{1}{r^2}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t}\frac{\partial t}{\partial \theta} = (\frac{-1}{r^2})(\dot{r})(\dot{\theta}^{-1}) = (\frac{-1}{r^2})(\dot{r})(\frac{r^2}{h}) = -\frac{\dot{r}}{h}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial t}(\frac{\partial u}{\partial \theta})\frac{\partial t}{\partial \theta} = -\frac{\ddot{r}}{h}\dot{\theta}^{-1} = -\frac{\ddot{r}}{h}\frac{r^2}{h} = -\frac{\ddot{r}}{u^2h^2}$$

from which we obtain:

$$\ddot{r} = -u^2h^2\frac{\partial^2 u}{\partial \theta^2}$$

Substitution of these partial derivatives in (13) results in:

$$-u^2h^2\frac{\partial^2 u}{\partial \theta^2} - \frac{h^2}{r^3} = -\mu u^2$$

so that:

$$\frac{\partial^2 u}{\partial \theta^2} + u = \frac{\mu}{h^2} \tag{17}$$

This equation is equivalent to that of a mathematical pendulum, its solution is:

$$u = A\cos\theta + B$$
$$\frac{\partial u}{\partial \theta} = -A\sin\theta$$
$$\frac{\partial^2 u}{\partial \theta^2} = -A\cos\theta$$

We find:

$$u + \frac{\partial^2 u}{\partial \theta^2} = B = \frac{\mu}{h^2}$$

so that A becomes an arbitrary integration constant. In most textbooks we find the following expression that relates r to θ:

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \tag{18}$$

This expression results in circular orbits for $e = 0$, or elliptical orbits for $0 < e < 1$. To verify eq. (18) we evaluate r at the apo-apsis and the peri-apsis.

$$\begin{aligned} u(\theta = 0) &= \frac{1}{a(1-e)} = +A + B \\ u(\theta = \pi) &= \frac{1}{a(1+e)} = -A + B \end{aligned}$$

From which we get:

$$\begin{aligned} A &= \frac{e}{a(1-e^2)} \\ B &= \frac{\mu}{h^2} \end{aligned}$$

The length of the angular momentum vector h is:

$$2B = \frac{1}{a(1-e)} + \frac{1}{a(1+e)} = \frac{2}{a(1-e^2)}$$
$$B = \frac{1}{a(1-e^2)} = \frac{\mu}{h^2}$$

resulting in:

$$h = \sqrt{\mu a(1-e^2)}$$

which provides the length of the angular momentum vector.

2.8 Parabolic and hyperbolic orbits

So far we have demonstrated that circular and elliptic orbits appear, but in textbooks you also find that parabolic and hyperbolic orbits exist as a solution of the Kepler problem. A parabolic orbit corresponds to $e = 1$, and in a hyperbolic orbit $e > 1$. The parabolic orbit is one where we arrive with a total energy of zero at infinity, therefore it is also called the minimum escape orbit. Another option to escape the planet is to fly in a hyperbolic orbit, in this case we arrive with a positive total energy at infinity.

2.9 The vis-viva equation

Equation (18) contains all information to confirm Kepler's first and second law. We will now switch to an energy consideration of the Keplerian motion. Because of the conservation of momentum we can not allow that energy disappears over time. This agrees with what we observe in astronomy; planets and moons do not disappear on a cosmologic time scale (which is only true if we leave tidal dissipation out of the discussion). If we assume that the total energy of the system is conserved then:

$$\frac{1}{2}mv^2 - \frac{m\mu}{r} = d^*$$

where m and v represent mass and scalar velocity and where d^* is constant. We eliminate the mass term m by considering $d = d^*/m$ so that:

$$\frac{v^2}{2} = d + \frac{\mu}{r}$$

The question is now to find d, since this would give us a relation to couple the scalar velocity in an orbit to the radius r. This is what we call the vis-viva equation or the path-speed equation.

At the peri-apsis and the apo-apsis the velocity vectors are perpendicular to r. The length of the moment vector (h) is nothing more than the product of the peri-apsis height and the corresponding scalar velocity v_p. The same property holds at the apo-apsis so that:

$$a(1-e)v_p = a(1+e)v_a \tag{19}$$

The energy balance at apo-apsis and peri-apsis is:

$$v_a^2 = 2d + 2\frac{\mu}{r_a} = 2d + 2\frac{\mu}{a(1+e)} \tag{20}$$

$$v_p^2 = 2d + 2\frac{\mu}{r_p} = 2d + 2\frac{\mu}{a(1-e)} \tag{21}$$

From equation (19) it follows that:

$$v_p^2 = \left(\frac{1+e}{1-e}\right)^2 v_a^2 \tag{22}$$

This equation is substituted in (21):

$$\left(\frac{1+e}{1-e}\right)^2 v_a^2 = 2d + 2\frac{\mu}{a(1-e)} \tag{23}$$

From this last equation and (20) it follows that:

$$v_a^2 = \left(\frac{1-e}{1+e}\right)^2 \left(2d + 2\frac{\mu}{a(1-e)}\right) = \left(2d + 2\frac{\mu}{a(1+e)}\right) \tag{24}$$

and as a result:

$$d = -\frac{\mu}{2a}$$

Summary:

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

and the vis-viva (Lat: path-velocity) relation becomes:

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

which is an important relation between r and v.

2.10 Orbital periods

For an circular orbit with $e = 0$ and $r = a$ we find that:

$$v = \sqrt{\frac{\mu}{a}}$$

If $v = na$ where n is a constant in radians per second then:

$$na = \sqrt{\frac{\mu}{a}} \Rightarrow \mu = n^2 a^3$$

This demonstrates Kepler's third law. Orbital periods for any parameter $e \in [0, 1]$ are denoted by τ and follow from the relation:

$$\tau = \frac{2\pi}{n} \;\Rightarrow\; \tau = 2\pi\sqrt{\frac{a^3}{\mu}}$$

The interested reader may ask why this is the case, why do we only need to calculate the orbital period τ of a circular orbit and why is there no need for a separate proof for elliptical orbits. The answer to this question is already hidden in the conservation of angular momentum, and related to this, the equal area law of Kepler. In an elliptical orbit the area dA of a segment spent in a small time interval dt is (due to the conservation of angular momentum) equal to $dA = \frac{1}{2}h$. The area A within the ellipse is:

$$A = \int_{\theta=0}^{2\pi} \frac{1}{2} r(\theta)^2 d\theta \tag{25}$$

To obtain the orbital period τ we try to fit the small segments dA within A, and we get:

$$\tau = A/dA = \int_{\theta=0}^{2\pi} \frac{r(\theta)^2}{h}\, d\theta = \int_{\theta=0}^{2\pi} \dot{\theta}^{-1}\, d\theta = \frac{2\pi a^2}{\sqrt{\mu a}} \tag{26}$$

which is valid for $a > 0$ and $0 \le e < 1$. This proofs that the orbital period of circular and elliptical orbits follows exactly from Kepler's 3rd law.

2.11 Kepler's orbit in three dimensions

To position a Kepler orbit in a three dimensional space we need three additional parameters to position the angular momentum vector $\overline{H}$. The standard solution is to consider an inclination parameter I which is the angle between the positive z-axis of the Earth in a quasi-inertial reference system and $\vec{H}$. In addition we define the angle Ω that provides the direction in the equatorial plane of the intersection between the orbit plane and the positive inertial x-axis, Ω is also called the right ascension of the ascending node. The last Kepler parameter is called ω, which provides the position in the orbital plane of the peri-apsis relative to the earlier mentioned intersection line.

The choice of these parameters is slightly ambiguous, because you can easily represent the same Keplerian orbit with different variables, as has been done by Delauney, Gauss and others. In any case, it should always be possible to convert an inertial position and velocity in three dimension to 6 equivalent orbit parameters.

2.12 Time vs True anomaly

The variable θ in equation (18) does NOT linearly progress in time. In fact, this is already explained in Kepler's equal area law. Without any further proof we present a well known method to convert the true anomaly, hereafter called f, into a time t relative to the last peri-apsis transit t_0:

- The mean anomaly M is defined as $M = n.(t - t_0)$ where n is the mean motion in radians per second.
- The eccentric anomaly E is related to M via a transcendental relation: $M = E - e \sin E$.
- The goniometric relation $\tan\theta = \sqrt{1-e^2}\sin E/(\cos E - e)$ is used to complete the conversion.

The most interesting step is to solve the transcendental relation. A well known algorithm relies on a Picard iteration, an alternative is an analytical series expansion.

3 Exercises

1. Show that $U = \frac{1}{r}$ is a solution of $\Delta U = 0$
2. Derive the equations of motion for a satellite near two planets.
3. What is the orbital period of Jupiter at 5 times the radius of the Earth's.
4. Plot $r(\theta)$, $v(\theta)$ and the angle between $r(\theta)$ and $v(\theta)$ for $\theta \in [0, 2\pi]$ and for $e = 0.01$ and $a = 10000$ km for $\mu = 3.986 \times 10^{14}$ $m^3 s^{-2}$.
5. For an elliptic orbit the total energy is negative, for a parabolic orbit the total energy is zero, ie. it is the orbit that allows to escape from earth to arrive with zero energy at infinity. How do you parameterize parabolic orbits, how do you proof that they are a solution of the Kepler problem?

6. Explain why Kepler's theory works so well for most orbital mechanic problems.
7. Make a perspective drawing of the Kepler ellipse in 3D and explain all involved variables.
8. Design a problem to plot ground tracks for an arbitrary Kepler orbit, assume a constant Earth rotation speed at a sidereal rate.
9. Implement the equations of motion for the Kepler problem in MATLAB and verify the numerical solution of r and v against the analytical formulas.
10. Demonstrate with the help of MATLAB that the total energy is conserved for the Kepler problem.