Dynamics I

15-nov-2006

If there is one thing that we understand very well about our solar system, then it is the way planets, moons, comets and asteroids move around.

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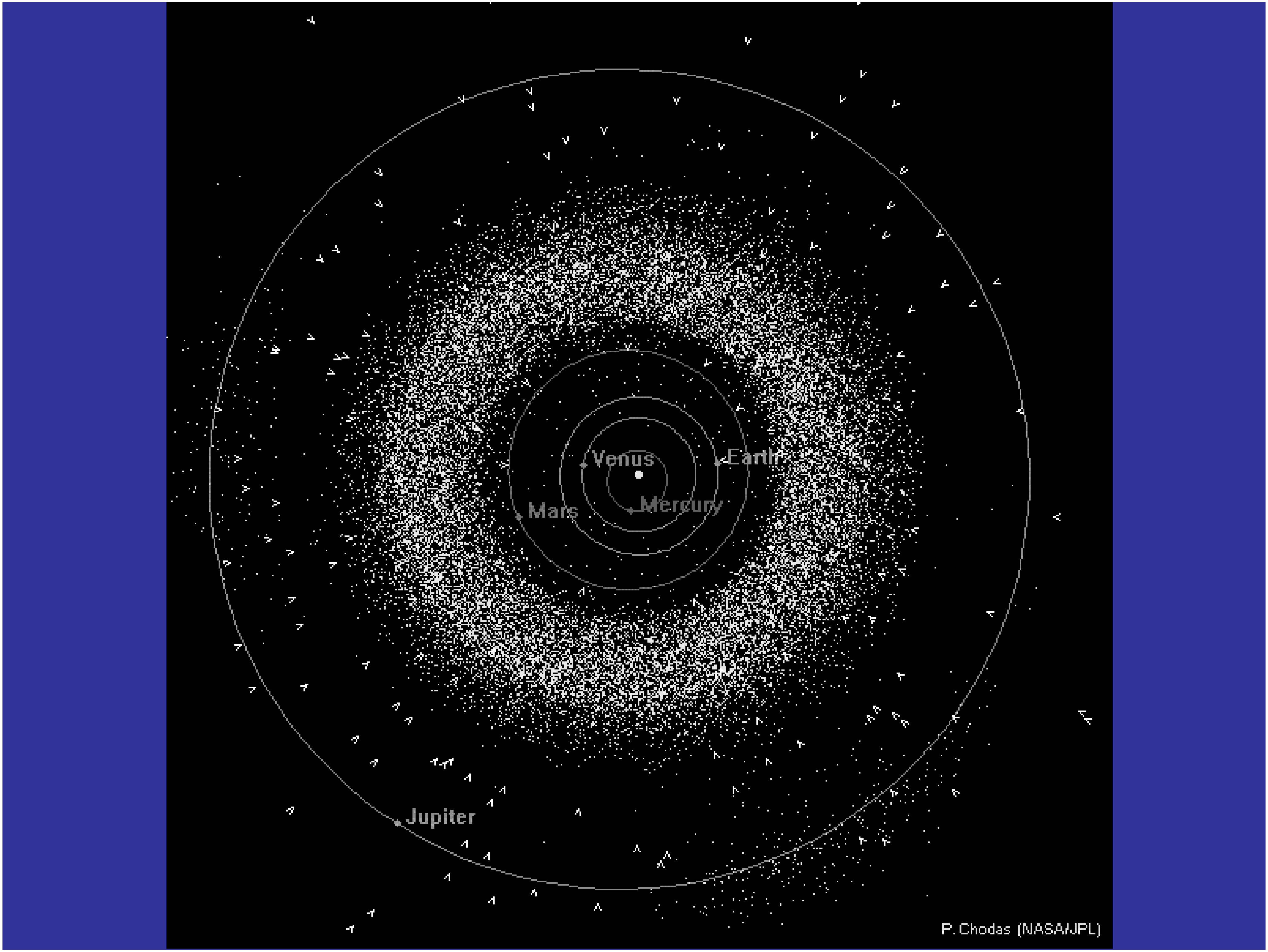

Venus
Earth
Mars
Mercury
Jupiter
P. Chodas (NASA/JPL)

Overview

– The two-body problem
– The three-body problem
– Hill equations
– ‘Planetary’ perturbations and resonances
– Long term stability of orbits
– Orbits about an oblate planet
– Tides in the solar system
– Dissipative forces and the orbits of small particles

The two-body problem

- What defines the problem?
  - A large planet and a smaller satellite
- Different views on the solar system
  - Nicolaus Copernicus
  - Tycho Brahe
  - Johannes Kepler
- Kepler’s laws on orbit motions
  - Elliptical orbits within an orbital plane
  - Equal area law
  - Scale vs orbital period law
- Equations of Motion

Copernicus, Brahe and Kepler

- In the 16th century, the Polish astronomer Nicolaus Copernicus replaced the traditional Earth-centered view of planetary motion with one in which the Sun is at the center and the planets move around it in circles.
- Although the Copernican model came quite close to correctly predicting planetary motion, discrepancies existed.
- This became particularly evident in the case of the planet Mars, whose orbit was very accurately measured by the Danish astronomer Tycho Brahe
- The problem was solved by the German mathematician Johannes Kepler, who found that planetary orbits are not circles, but ellipses.
- Johannes Kepler described the observed planetary motions with the three well-known Keplerian laws.

Keplerian Laws

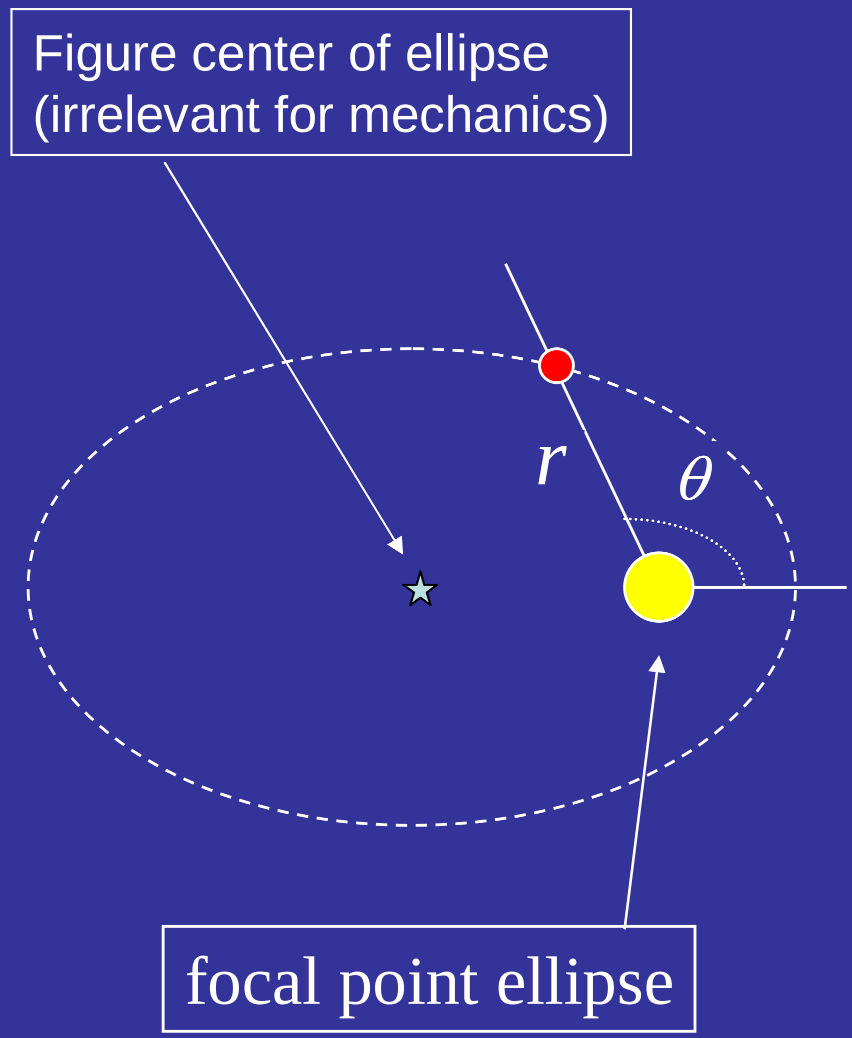
- ***Law I***
  - Each planet revolves around the Sun in an elliptical path, with the Sun occupying one of the foci of the ellipse.
- ***Law II***
  - **The straight line joining the Sun and a planet sweeps out equal areas in equal intervals of time.**
- ***Law III***
  - **The squares of the planets' orbital periods are proportional to the cubes of the semi-major axes of their orbits.**

Kepler's first law

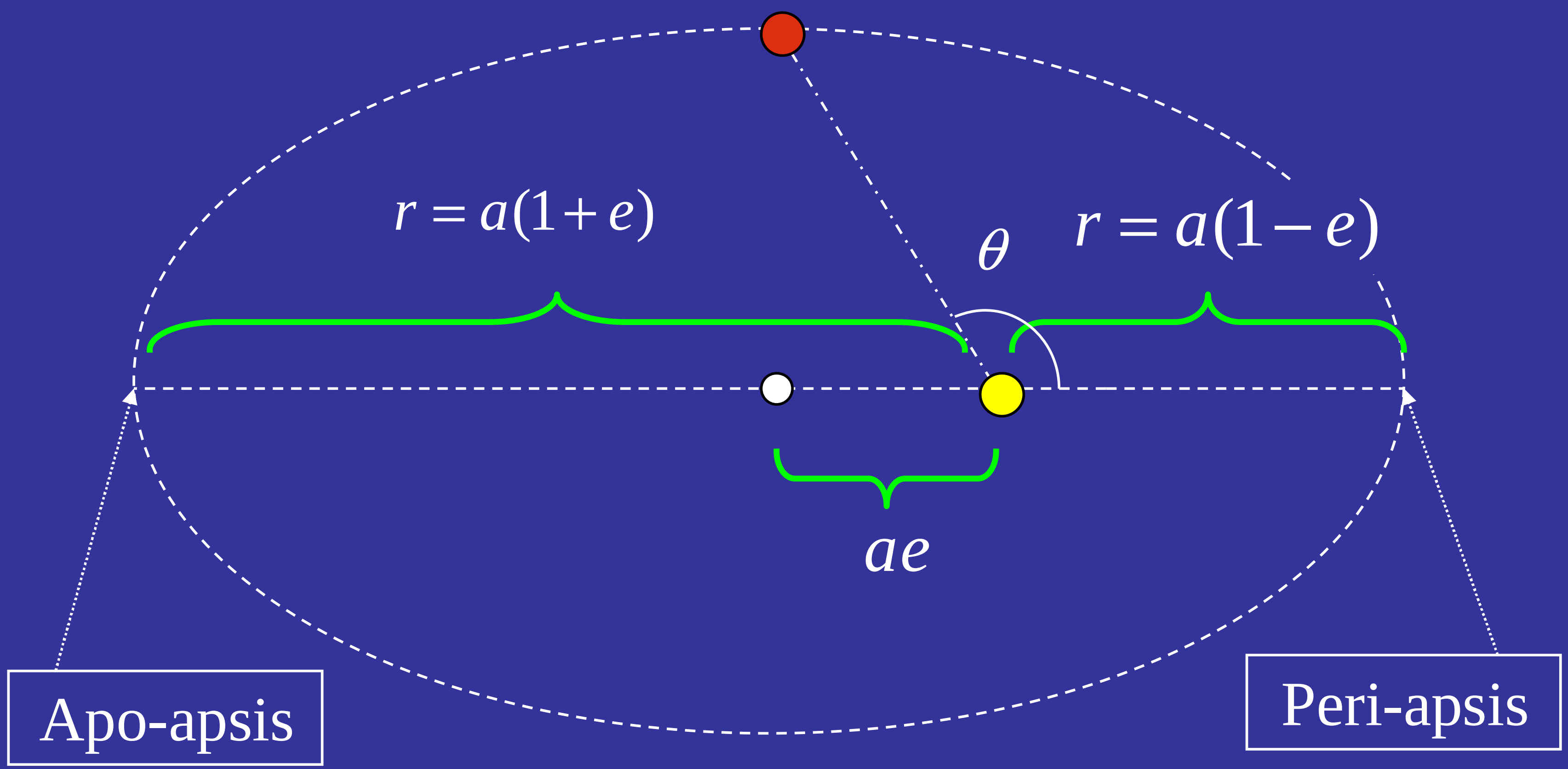
$$r(\theta) = \frac{a(1-e^2)}{1+e\cos(\theta)}$$

There are 4 cases:

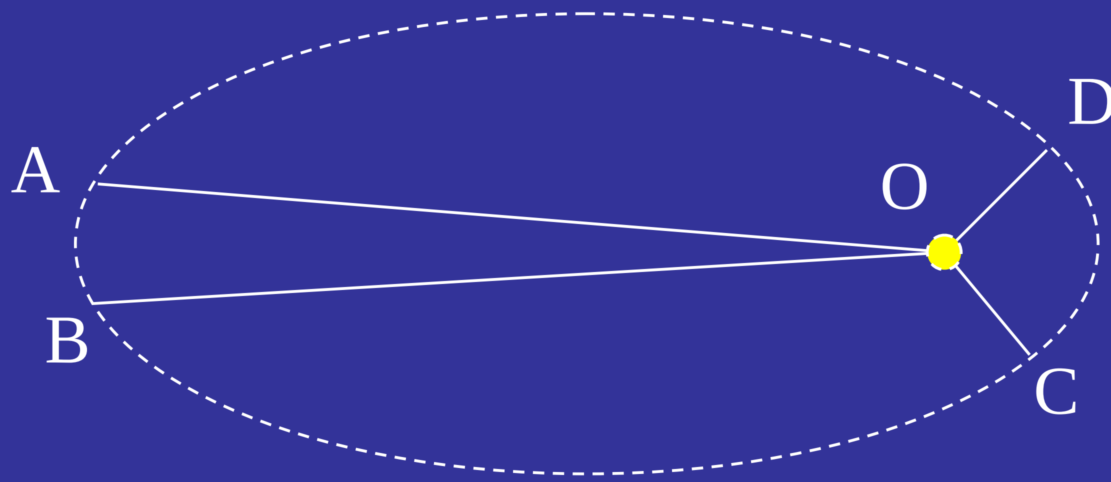
- e=0, circle
- 0<e<1, ellipse
- e=1, parabola
- e>1, hyperbola



In-plane Kepler parameters



Kepler’s second law



$ABO \equiv CDO$

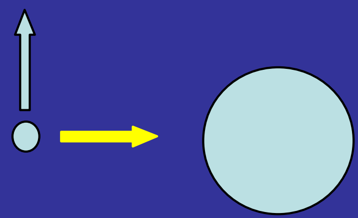
Kepler's Third law

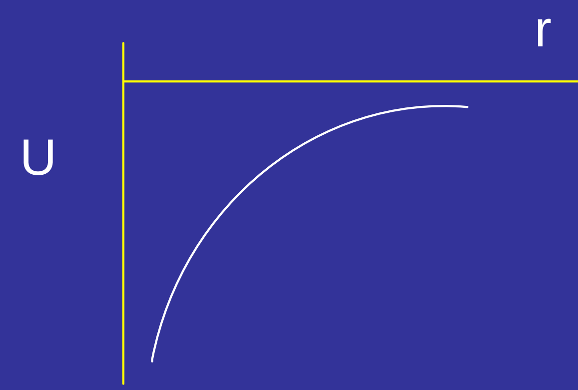
$$n^2 a^3 = G(M + m) \qquad\qquad T = \frac{2\pi}{n}$$

The variable *n* represents the mean motion in radians per second, *a* is the semi-major axis, *G* is the gravitational constant, *M* is the mass of the "Sun", *m* is the mass of the satellite ($m << M$) and *T* is the orbital period of the satellite

In astronomy this law provides the scale of the Solar System, we can observe rather precisely the orbital periods of planets, and from this information we can infer the scale of the solar system. Everything is then normalized to the Earth's orbital radius, which is said to be 1 astronomical unit (1 AU)

Equations of motion





$$\ddot{\bar{x}} = -\nabla U(\bar{x})$$

$$U(\bar{x}) = -\frac{GM}{r} = -\frac{\mu}{r}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\bar{x} = (x, y, z)^T$$

$$\Rightarrow \quad \ddot{\bar{x}} = -\frac{\mu}{r^3}\bar{x}$$

These equations hold in the inertial coordinate frame and they are only valid for the Kepler problem

Potential theory

- U is known as the potential, it is equivalent to the potential energy of an object scaled to its mass
- By definition U is zero at infinity
- The Laplacian of U is zero ($\Delta U=0$) outside to mass that it generating the potential ★
- The gradient of U is what we call gravity
- $U=-\mu/r$ is an approximation of $\Delta U=0$ ★
- A more complete solution of U uses so-called spherical harmonic functions

Why is there an orbital plane, why no other motion?

From mechanics we know that

$$\overline{H} = \overline{X} \times \dot{\overline{X}}$$

where $\overline{H}$ is the angular momentum vector. Differentiation to time and substitution of the equations of motion gives:

$$\frac{\partial \overline{H}}{\partial t} = \frac{\partial \overline{X} \times \dot{\overline{X}}}{\partial t} = \dot{\overline{X}} \times \dot{\overline{X}} + \overline{X} \times \ddot{\overline{X}} \Rightarrow$$

$$\overline{X} \times \ddot{\overline{X}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} -x.GM/r^3 \\ -y.GM/r^3 \\ -z.GM/r^3 \end{bmatrix} = -\frac{GM}{r^3} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dot{\overline{H}} = \overline{0}$$

As a result $\overline{H}$ can not change in time and the motion is constrained to an orbital plane.

How to obtain r(θ)

- A particle moves in a central force field
- The motion takes place within an orbital plane
- The solution of the equation of motion is represented in the orbital plane
- Substitution 1: polar coordinates in the orbital plane
- Substitution 2: replace r by 1/u
- Analogy with a mathematical pendulum
- Solve this an substitute elliptical configuration
- Final step: transformation orbital plane to 3D (this gives us the set of 6 Keplerian elements)

Mathematics on r(θ):

$$\ddot{\bar{x}} = -\nabla V + \cancel{\sum_i \bar{f}^i}$$

$$V(r) = -\frac{\mu}{r}$$

$$\underbrace{\qquad\qquad\qquad}$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$$\Downarrow$$

Essential information ->

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

No new information ->

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

Mathematics on r(θ):

Substitute:

$$r = 1/u$$

$$\frac{\partial^2 u}{\partial \theta^2} + u = \frac{\mu}{h^2}$$

(h = length ang mom vector)

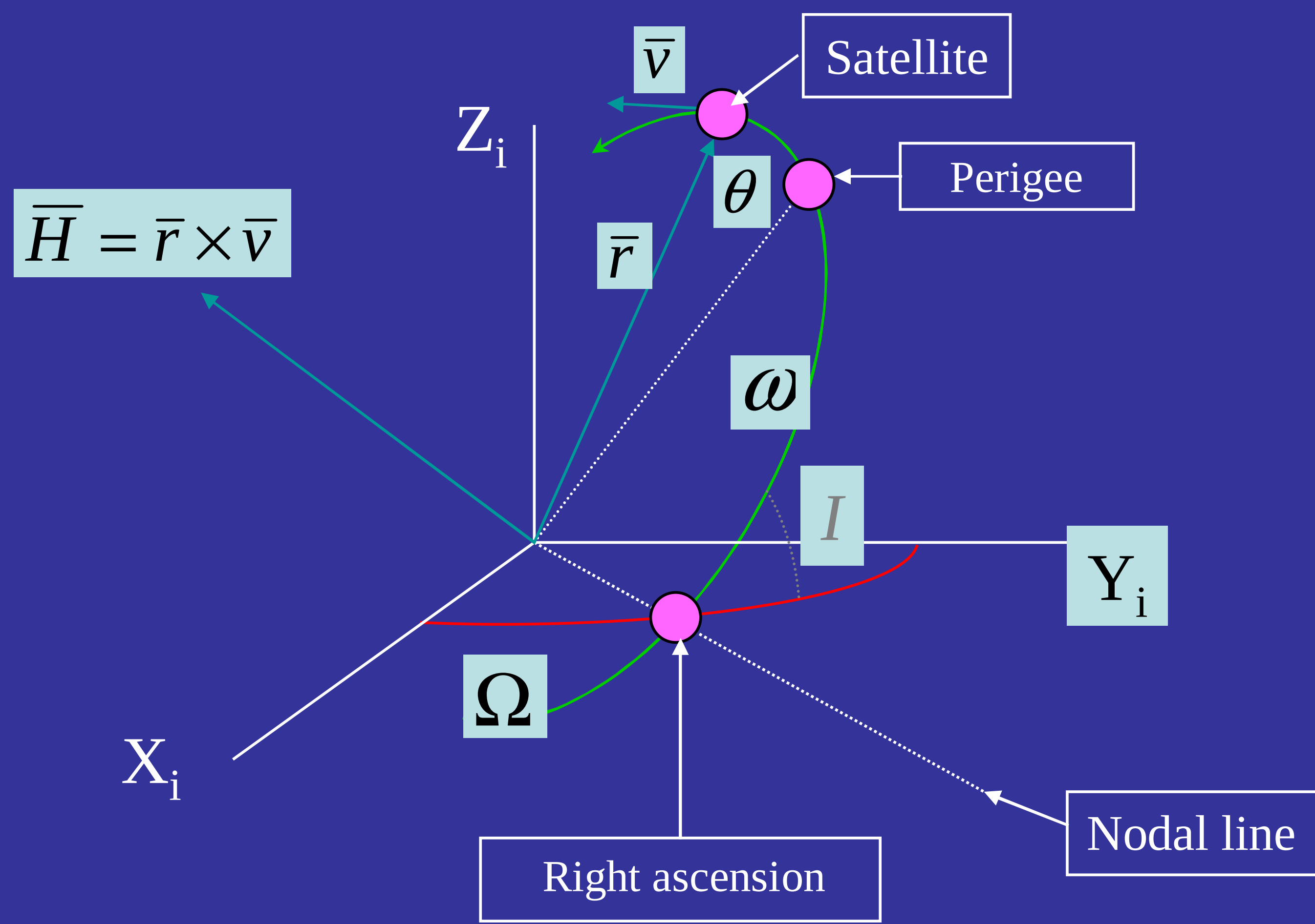
Characteristic:

$$u = A\cos\theta + B$$

$$\begin{aligned} A &= \frac{e}{a(1-e^2)} \\ B &= \frac{\mu}{h^2} \end{aligned}$$

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$

Kepler's solution in an inertial coordinate system



XYZ: inertial cs

Ω: right ascension

ω: argument van perigee

θ: true anomaly

I: Inclination orbit plane

H: angular momentum vector

r: position vector satellite

v: velocity satellite

Velocity and Position (aka vis-viva equations)

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = a(1-e.\cos E)$$

radius r

$$v = \sqrt{GM\left(\frac{2}{r}-\frac{1}{a}\right)}$$

velocity v

Note: in this case only θ, or E or M depend on time.

Total Energy

- The total energy in the system is the sum of kinetic and potential energy
- For the Kepler problem one can show that the total energy is half that of the potential energy

$$E_{kin} + E_{pot} = E_{tot}$$

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = c.m \Rightarrow$$

$$c = -\frac{GM}{2a} \Rightarrow$$

Potential Energy at $a = 2c$

Kepler's equation

- *There is a difference between the definition of the true anomaly θ, the eccentric anomaly E and the mean anomaly M*
- *Note: do not confuse E and the eccentricity parameter e*

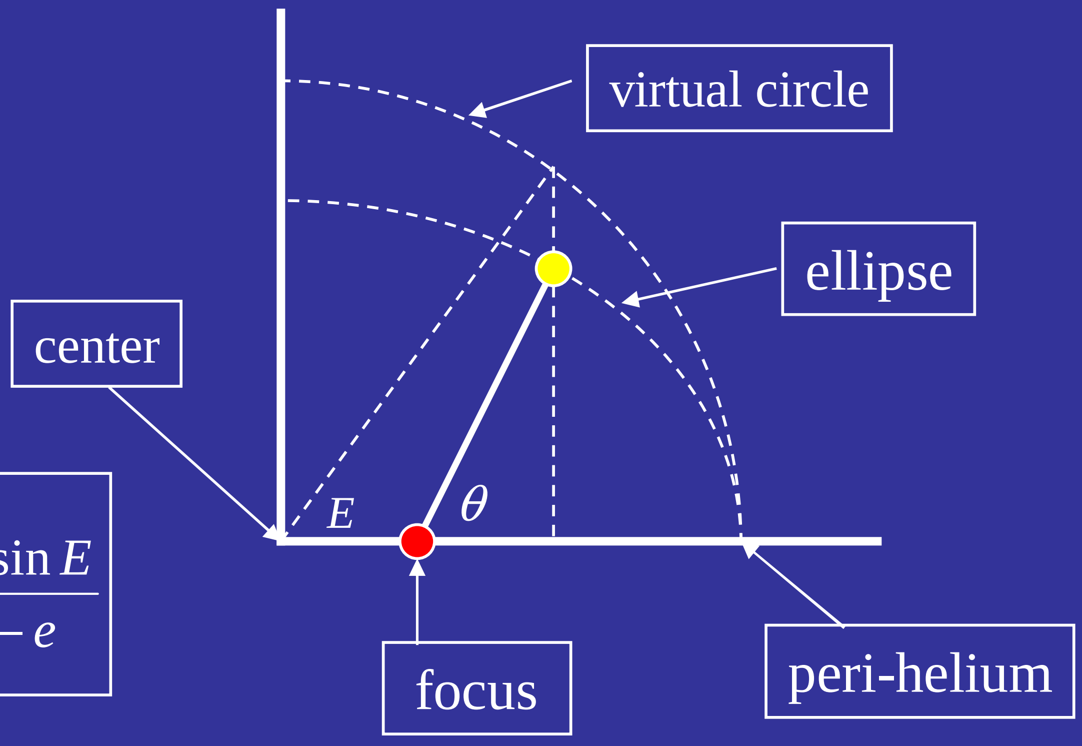
Transcendental relation:

$$M = E - e \sin(E)$$

$$M = n\,(t - t_0)$$

This is Kepler's equation

Second relation: $\tan\theta = \dfrac{\sqrt{1-e^2}\,\sin E}{\cos E - e}$



Keplerian elements

Position and velocity follow from:

- The semi major axis ***a***
- The eccentricity ***e***
- The inclination of the orbital plane ***I***
- The right ascension of the ascending node Ω
- The argument van perigee ω
- Anomalistic angles in the orbit plane ***M***, ***E*** and θ

Also:

- You should be able to “draw” all elements
- Linear combinations of elements are often used
- Non-singular elements (Gauβ, Delaunay, Hill)

Example problem 1

Situation:

The Earth has a semi major axis at 1 AU and e=0.01

Question 1:

what are the values of r in the peri-helium and apo-helium

Question 2:

what is the orbital period for a planet at a=1.5 AU and e=0.02

Question 3:

plot for the Earth a graph of r as a function of the true anomaly θ

Question 4:

what is the escape velocity from Earth?

Example problem 2

- Jupiter is at about 5 AU, and a body from the asteroid belt (beyond Mars, but within Jupiter) performs a flyby at Jupiter
  - Configuration 1: What ΔV is required to escape the solar system?
  - Configuration 2: What ΔV is required for an orbit with a peri-helium smaller than 1 AU?
  - Show that configuration 2 occurs less often than configuration 1 (In other words, how much percent swings into the solar system, how disappears?)

Example problem 3

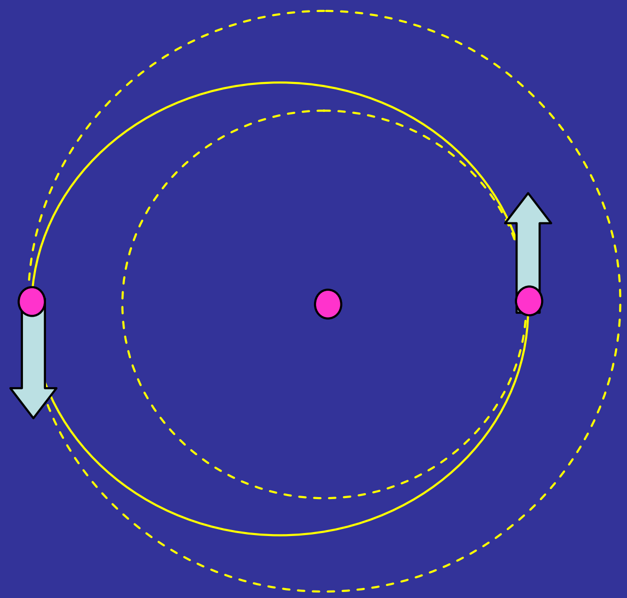
- Show that the total energy is conserved for a particle in a Kepler orbit
- Hints:
  - Compute the kinetic energy at periapsis
  - Compute the potential energy at periapsis
  - Why is it sufficient to calculate the sum at one point in orbit?
  - Hint: consider the Laplacian of U

Example problem 4

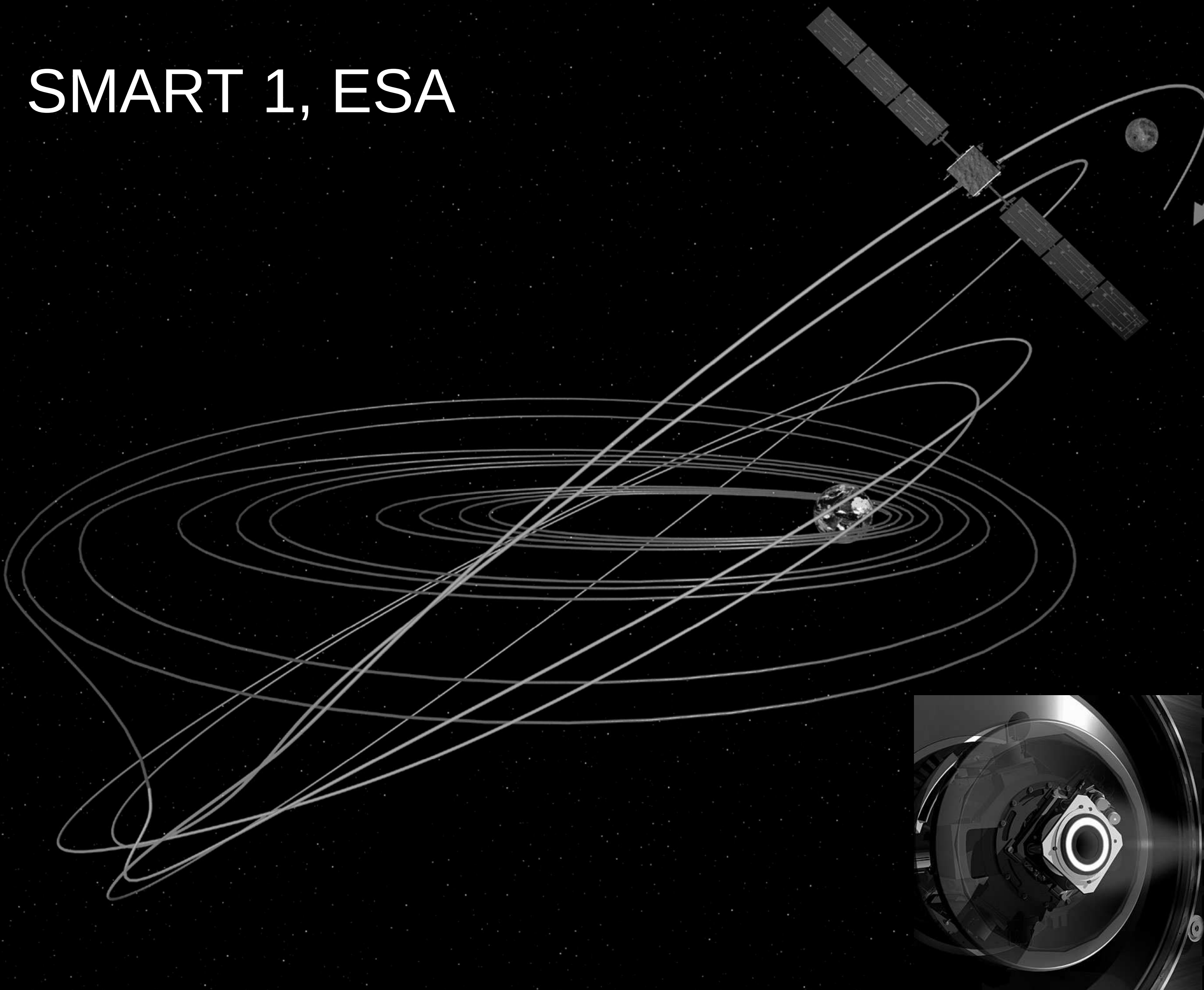
- Show that U=1/r is consistent with Newton's gravitational law
- Hints:
  - acceleration = force / mass
  - acceleration = gradient of U
- Related to this problem
  - Why is the Laplacian of U (a.k.a. ΔU) in 3D equal to 0 for the exterior.

Example problem 5

- Hohmann orbits classify as a transfer trajectory from two circular orbits with radii r_1 and $r_{2.}$
- To enter the transfer orbit we apply ΔV_1 and we we arrive we apply ΔV_2
- What is total ΔV to complete the transfer?
- What is the ratio between ΔV_1 and ΔV_2



SMART 1, ESA

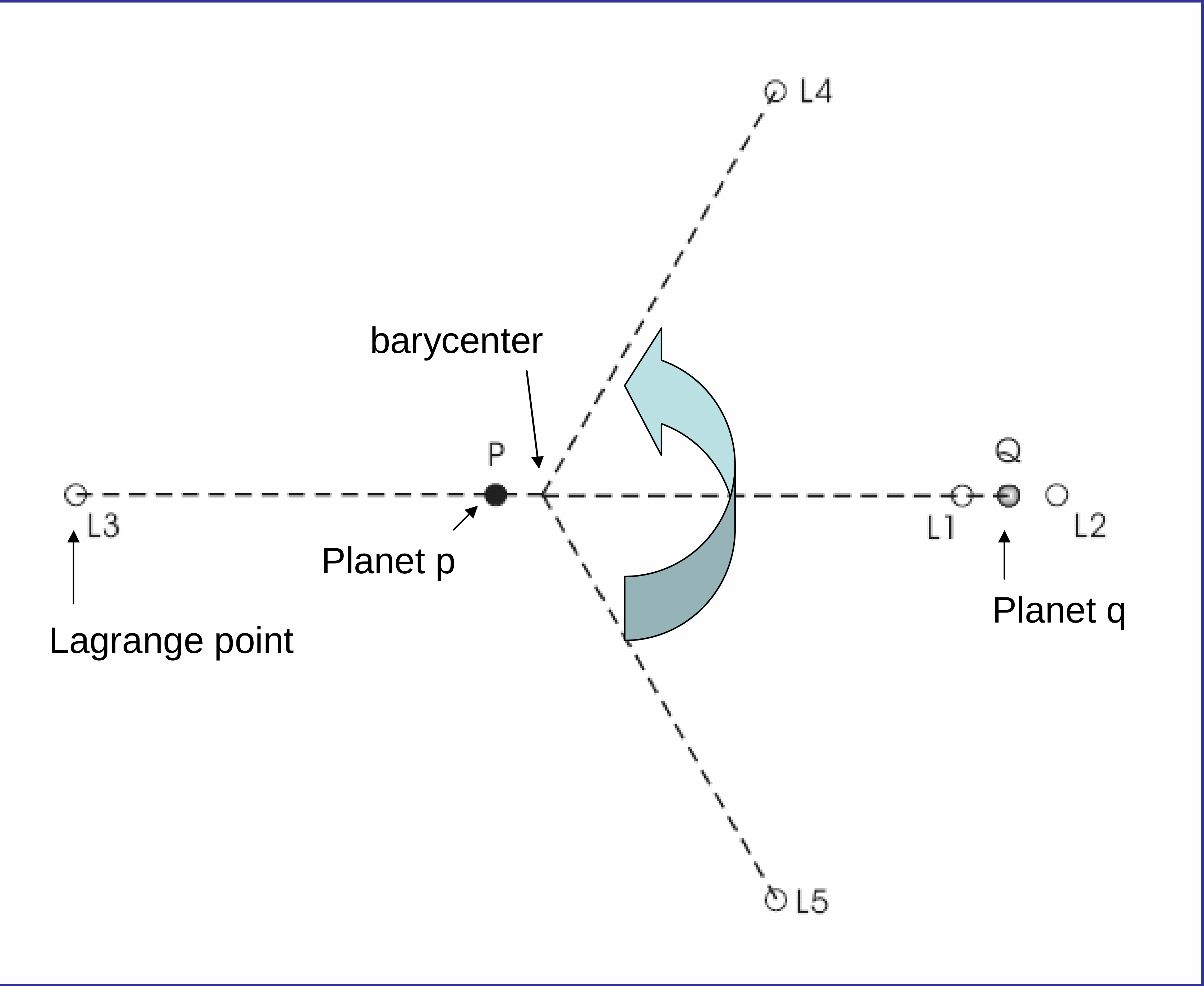


The three body problem

- Configuration consists of three arbitrary masses that attract one another
- Take Newton’s gravity Law and add up all the forces (and convert to accelerations)
- Define the barycenter of the system
- Only numerical solutions are tractable
- More common in astronomy is the “restricted three body problem”

Restricted three body problem

- Configuration consists of two large masses (m_p and m_q) that are about of the same size.
- In addition there is a small particle
- The barycenter is between m_p and m_q
- The system is rotating at a uniform rate
- Equations of motion include “frame” terms as a result of this rotation



See also: http://janus.astro.umd.edu/javadir/orbits/ssv.html

Balance rotation and gravity

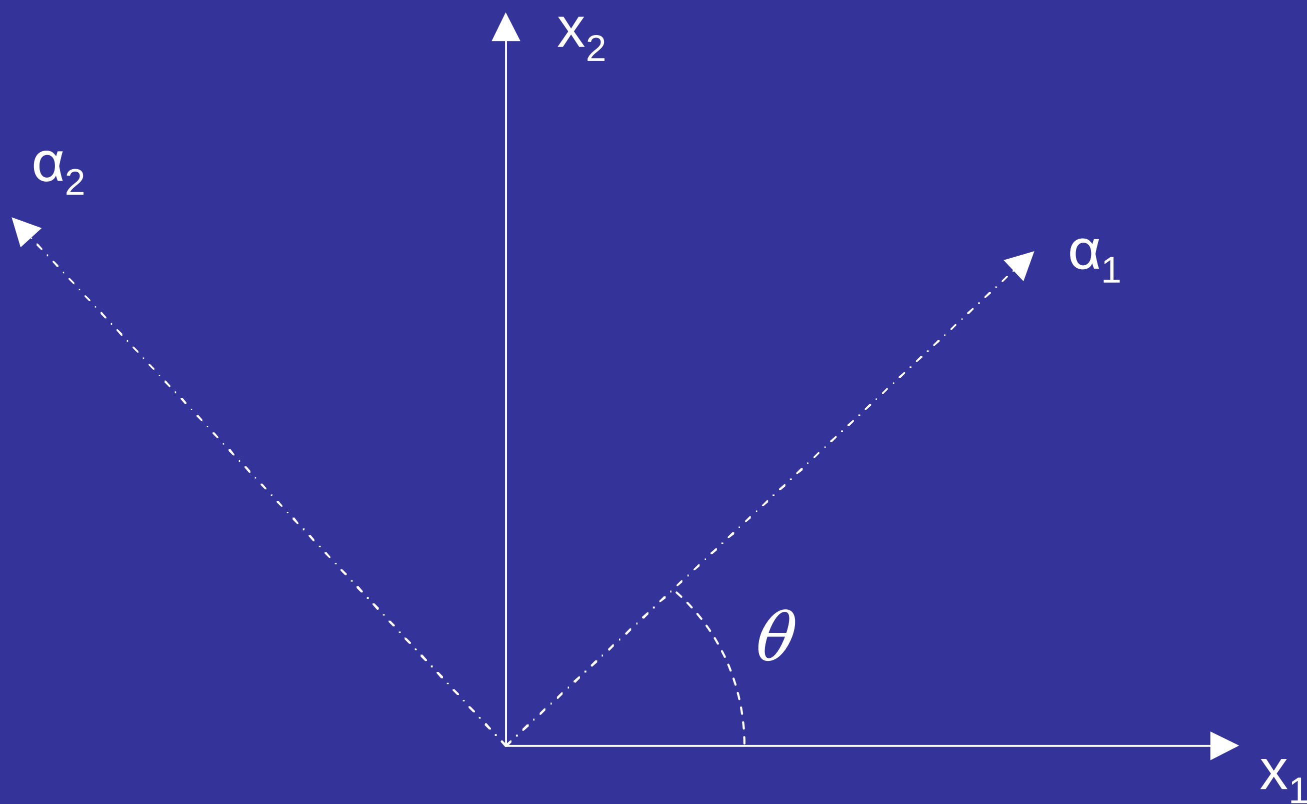
For m_p and m_q we have:

$$n^2 d_p = \frac{1}{m_p}\frac{G m_p m_q}{(d_p + d_q)^2} = \frac{\mu_q}{(d_p + d_q)^2}$$

$$n^2 d_q = \frac{1}{m_q}\frac{G m_p m_q}{(d_p + d_q)^2} = \frac{\mu_p}{(d_p + d_q)^2}$$

$$n^2 = \frac{\mu_p + \mu_q}{(d_p + d_q)^3}$$

Uniform rotation



$$\overline{x} = R_3(\theta)\,\overline{\alpha}$$

Equations of motion after rotation

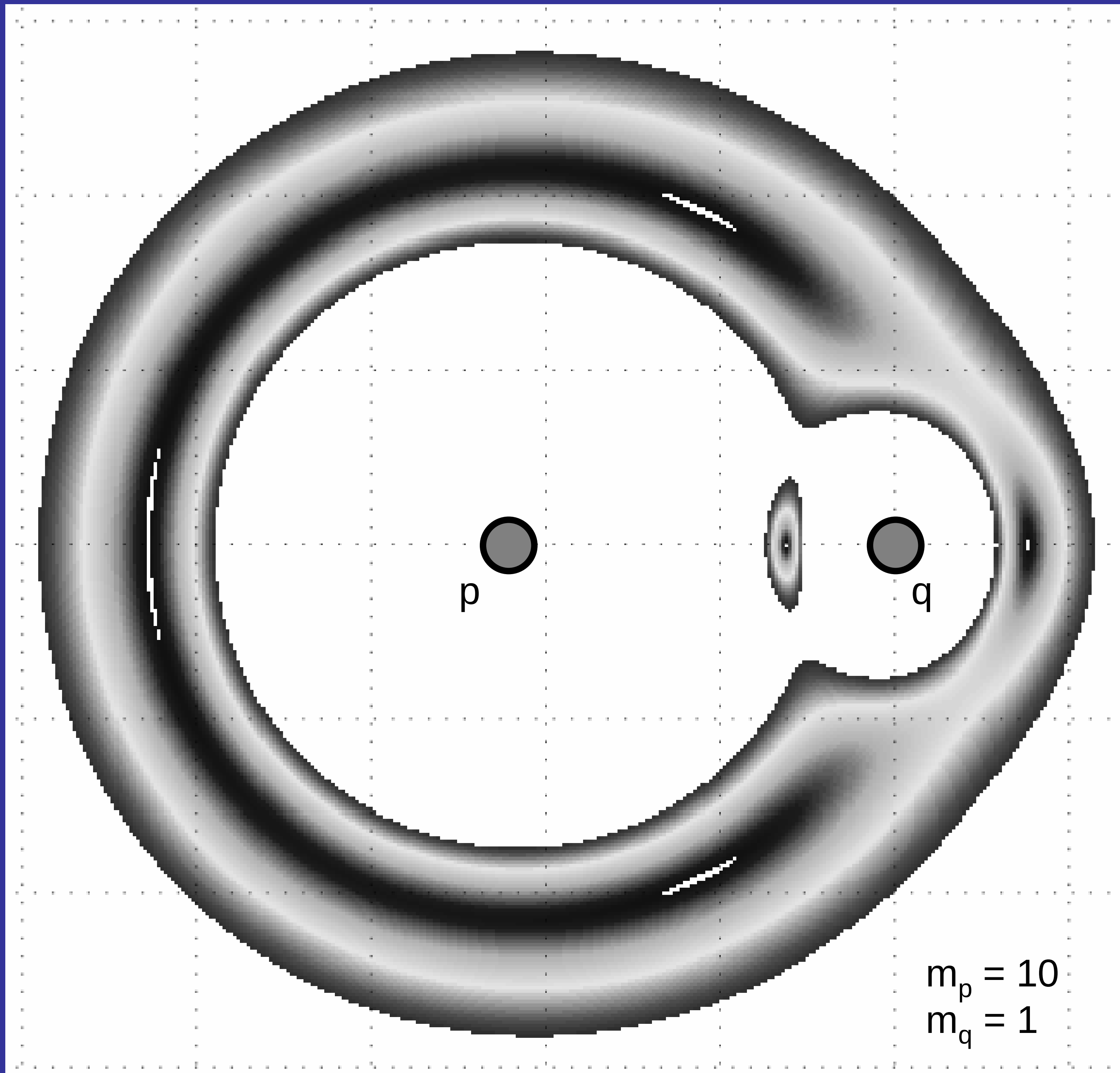
After straightforward differentiation we get

$$\ddot{\overline{x}} = \begin{bmatrix} \ddot{\alpha}_1 - 2n\dot{\alpha}_2 - n^2\alpha_1 \\ \ddot{\alpha}_2 + 2n\dot{\alpha}_1 - n^2\alpha_2 \\ \ddot{\alpha}_3 \end{bmatrix} = \frac{-\mu_p}{|\overline{\alpha} - \overline{\alpha}_p|^3}(\overline{\alpha} - \overline{\alpha}_p) + \frac{-\mu_q}{|\overline{\alpha} - \overline{\alpha}_q|^3}(\overline{\alpha} - \overline{\alpha}_q)$$

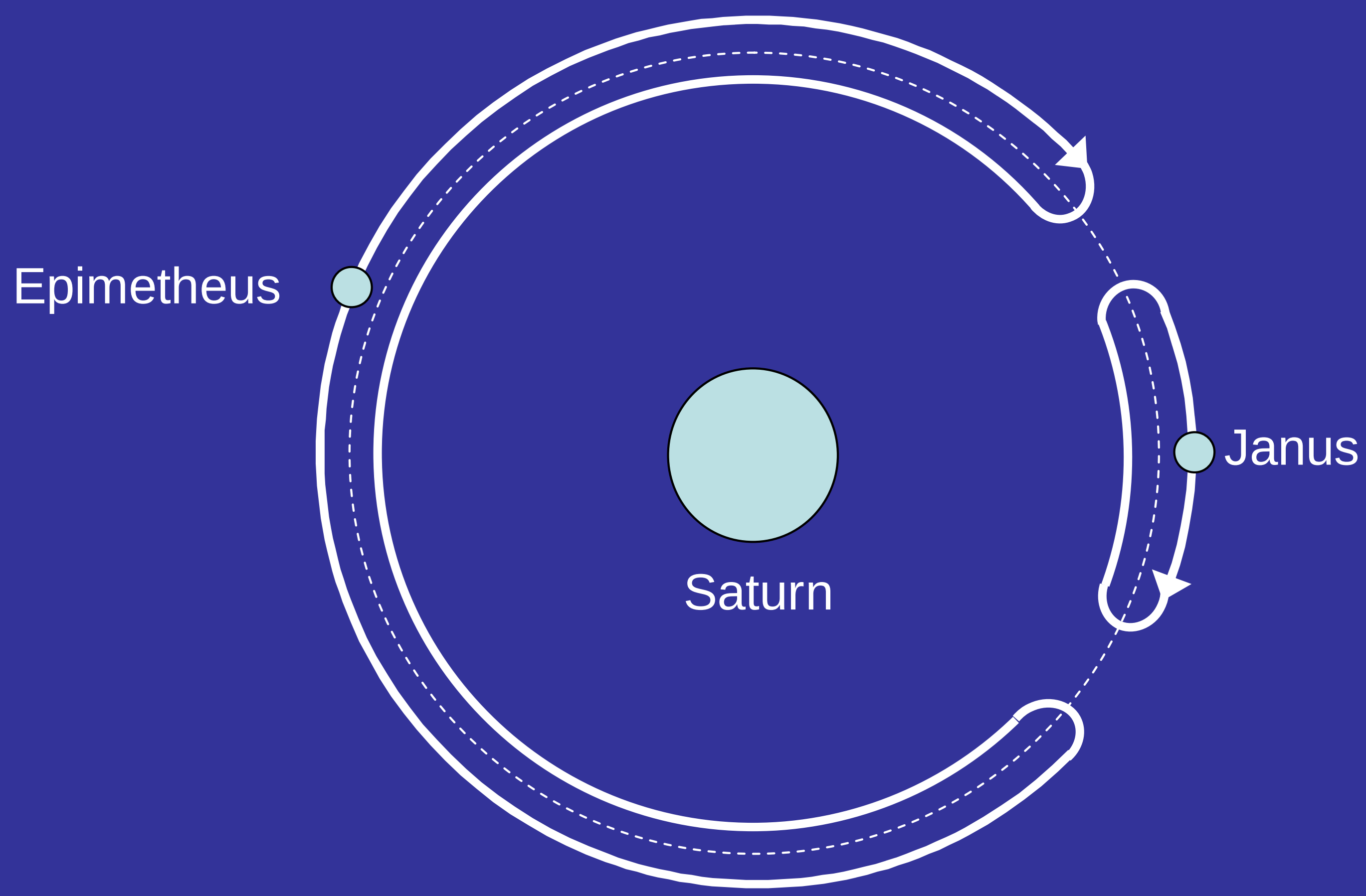
If we ignore the Coriolis term, then we obtain

$$\begin{aligned} \ddot{\alpha}_1 &= -\mu_p \frac{\alpha_1 + d_p}{|\overline{\alpha} - \overline{\alpha}_p|^3} - \mu_q \frac{\alpha_1 - d_q}{|\overline{\alpha} - \overline{\alpha}_q|^3} - n^2\alpha_1 \\ \ddot{\alpha}_2 &= -\mu_p \frac{\alpha_2}{|\overline{\alpha} - \overline{\alpha}_p|^3} - \mu_q \frac{\alpha_2}{|\overline{\alpha} - \overline{\alpha}_q|^3} - n^2\alpha_2 \end{aligned}$$

So that we can plot the length of the acceleration vector on the left hand side to demonstrate the existence of the Lagrangian points L1 till L5 →


p
q
$m_p = 10$
$m_q = 1$

Horseshoe and Tadpole orbits



Example problem 6

- Identify all Lagrange points on the previous slide
- Show that L4 and L5 are at the locations where we find them
- Compute the positions of L1 to L3.