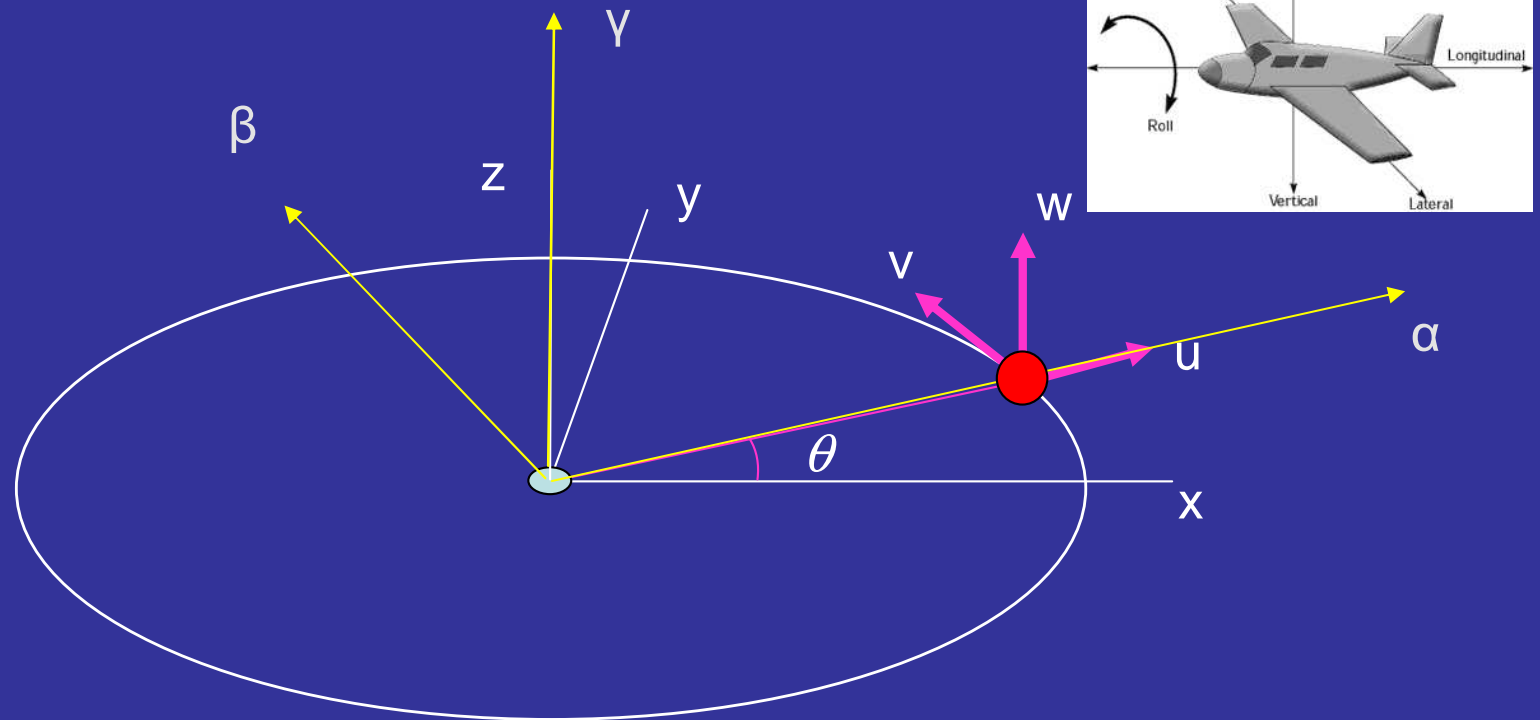# Dynamics II

17-nov-2006

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Course Contents Dynamics 2

- Hill equations
- Hill sphere
- Planetary perturbations and resonances
- Tides
  - Forcing
  - Roche limit
  - Dissipation
- Other small accelerations

Hill equation set-up

$$\overline{x} = R(\theta)\overline{\alpha} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\theta(t) = \theta_0 + \dot{\theta}t = \theta_0 + nt$$

Some intermediate steps

For the equations of motion we always need the acceleration and the gradient of the potential (V)

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = R(\theta) \begin{bmatrix} \ddot{\alpha} - 2n\dot{\beta} - n^2\alpha \\ \ddot{\beta} + 2n\dot{\alpha} - n^2\beta \\ \ddot{\gamma} \end{bmatrix}$$

$$\begin{bmatrix} \partial V/\partial x \\ \partial V/\partial y \\ \partial V/\partial z \end{bmatrix} = R(\theta) \begin{bmatrix} \partial V/\partial \alpha \\ \partial V/\partial \beta \\ \partial V/\partial \gamma \end{bmatrix}$$

$$\ddot{\alpha} - 2n\dot{\beta} - n^2\alpha = \frac{\partial V}{\partial \alpha}$$

$$\ddot{\beta} + 2n\dot{\alpha} - n^2\beta = \frac{\partial V}{\partial \beta}$$

$$\ddot{\gamma} = \frac{\partial V}{\partial \gamma}$$

(aka unfinished Hill equations)

Potential function V=U+T

- The U part belongs to a point mass (U=μ/r)
- T is called the disturbing potential
- We approximate V at the position of the satellite, thus (r+u,v,w) in the rotated frame:

$$\frac{\partial V}{\partial \alpha} = -\frac{\mu}{r^2} + 2\frac{\mu}{r^3}u + \frac{\partial T}{\partial u} + \cdots$$

$$\frac{\partial V}{\partial \beta} = -\frac{\mu}{r^3}v + \frac{\partial T}{\partial v} + \cdots$$

$$\frac{\partial V}{\partial \gamma} = -\frac{\mu}{r^3}w + \frac{\partial T}{\partial w} + \cdots$$

This last relation is now substituted in the "unfinished version of the Hill equations"

Hill equations

- For the rotating coordinate system one can show that:

$$\ddot{u} - 2n\dot{v} - 3n^2 u = \frac{\partial T}{\partial u}$$

$$\ddot{v} + 2n\dot{u} = \frac{\partial T}{\partial v}$$

$$\ddot{w} + n^2 w = \frac{\partial T}{\partial w}$$

- Here u, v and w are local coordinates for a satellite relative to a circular reference orbit (thus, u: radial, v: flight direction and w: normal to plane, and T is a perturbing potential whose partial derivates are required in the local co-rotating frame)

Solutions of the Hill Equations

- Homogeneous:

$$\dot{\overline{u}} = F\overline{u} \implies$$

$$F = Q\Lambda Q^T \implies$$

$$\overline{u}(t) = Q e^{\Lambda.(t-t_0)} Q^T \overline{u}(t_0)$$

- Particular:

$$\dot{\overline{u}} + F\overline{u} = \overline{b}(t) \quad \text{with} \quad F = Q\Lambda Q^T \implies$$

$$Q^T(\dot{\overline{u}} + F\overline{u}) = Q^T \overline{b}(x) \implies \dot{\overline{z}} + \Lambda\overline{z} = \overline{c}(t)$$

Solutions of the Hill Equations (2):

For each ith equation in the reduced particular system you should apply a Laplace transform:

$$\Phi_i(t) = d_i\,e^{-\lambda_i t} + e^{-\lambda_i t}\,C_i(t)$$

$$C_i(t) = \int_{t_0}^{t} e^{\lambda_i x}\,c_i(x)\,dx$$

$$u_i(t) = Q\,\Phi_i(t)$$

Characteristic solution homogenous case

$$\ddot{u} - 2n\dot{v} - 3n^2 u = 0$$

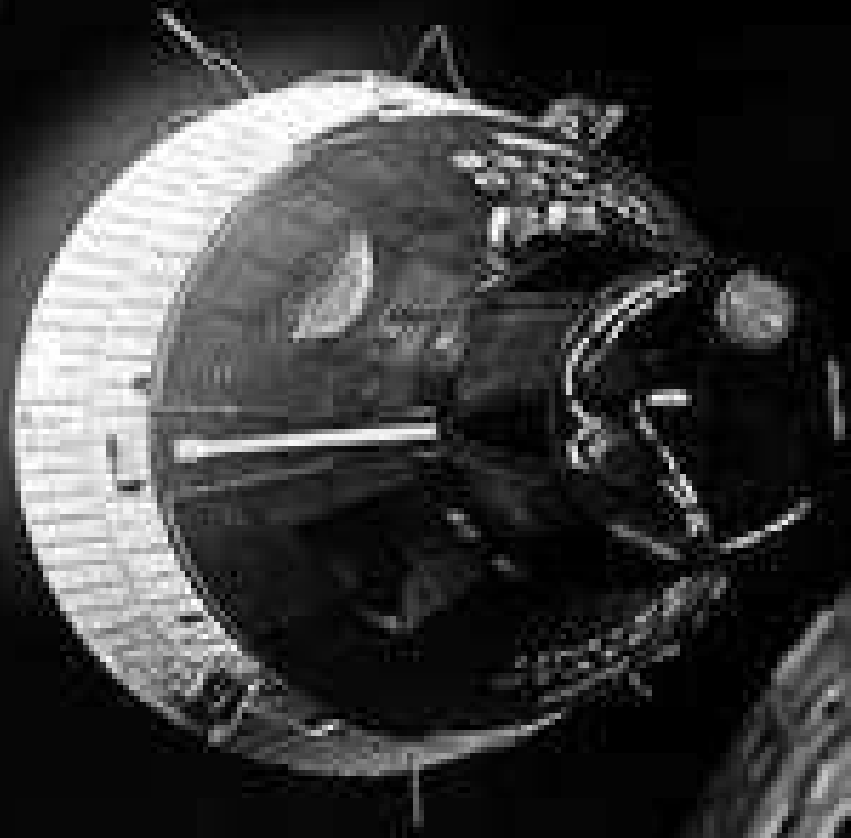$$\ddot{v} + 2n\dot{u} = 0$$

$$\ddot{w} + n^2 w = 0$$

$$u(t) = a_u \cos nt + b_u \sin nt + c_u$$

$$v(t) = a_v \cos nt + b_v \sin nt + c_v + d_v t$$

$$w(t) = a_w \cos nt + b_w \sin nt$$

This example works well for rendezvous and docking type op situations

Gemini 7, 1965

Characteristic solution particular case

$$\ddot{u} - 2n\dot{v} - 3n^2 u = P_u \cos \omega t + Q_u \sin \omega t$$
$$\ddot{v} + 2n\dot{u} = P_v \cos \omega t + Q_v \sin \omega t$$
$$\ddot{w} + n^2 w = P_w \cos \omega t + Q_w \sin \omega t$$

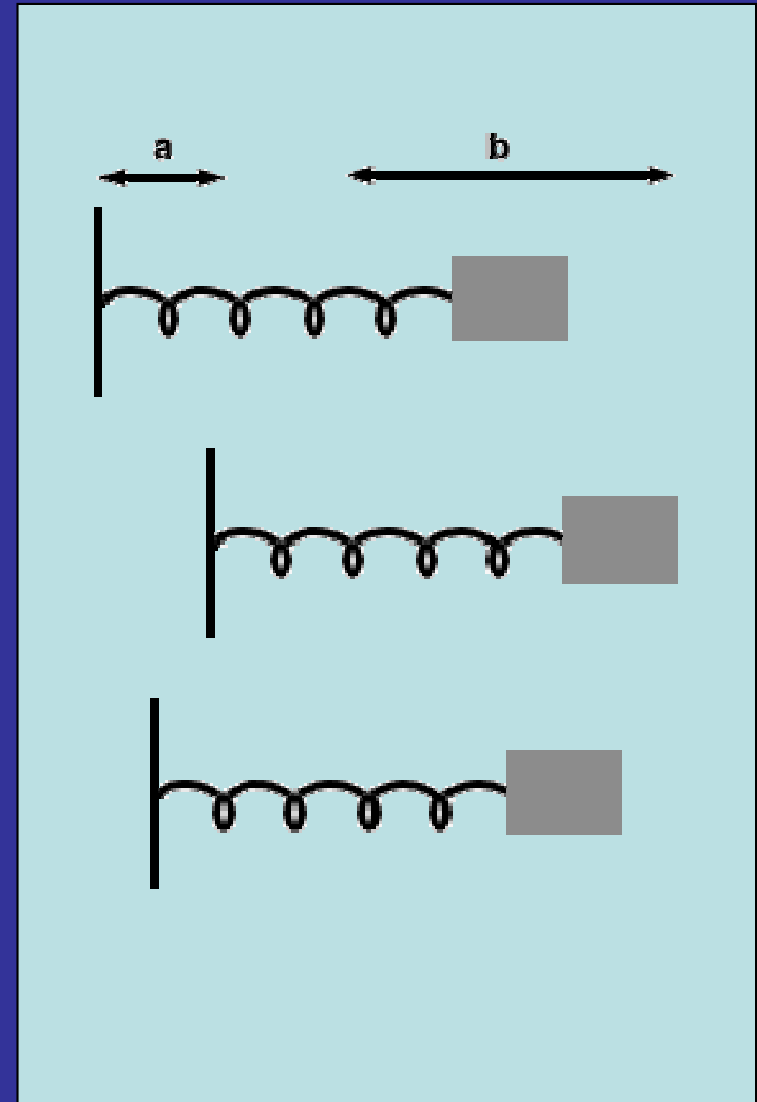$$u(t) = a_u \cos \omega t + b_u \sin \omega t$$
$$v(t) = a_v \cos \omega t + b_v \sin \omega t$$
$$w(t) = a_w \cos \omega t + b_w \sin \omega t$$

$$u(t) = \frac{-2nQ_v + \omega P_u}{\omega(n^2 - \omega^2)} \cos \omega t + \frac{2nP_v + \omega P_u}{\omega(n^2 - \omega^2)} \sin \omega t$$

Corollary

- We apply an acceleration function at frequency ω
- The orbit is perturbed at frequency ω
- Resonances appear when ω is near n
- It works in the same way as mass spring systems do

Example problem 1

- Derive the analytical solution of the homogeneous solution of the Hill equations, and substitute initial conditions
- Explain how you can solve a rendez vous between the Space shuttle and a Space Station with the obtained homogeneous solution of the Hill equation.

Hill Sphere

- The approximate limit to a secondary's (Moon's or planet's) gravitational dominance is given by the extent of its Hill sphere:

$$R_H \approx a \left(\frac{m_q}{3(m_p + m_q)} \right)^{1/3}$$

- A test particle located at the boundary of the Hill sphere is subject to a gravitational force from the planet comparable to the tidal difference between the force of the Sun on the planet and that on the test body
- The Hill sphere stretches out to the L1 Lagrange point
- All planetocentric orbits within the Hill sphere are stable over long periods of time, this is where we find all natural satellites.
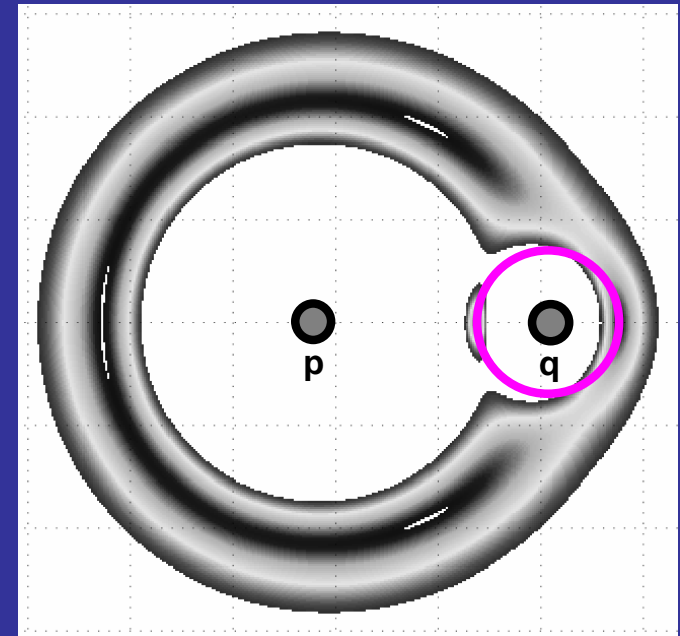
Why is R$_h$ like shown?

$$\ddot{u} - 2n\dot{v} - 3n^2u = \frac{\partial T}{\partial u}$$
$$\ddot{v} + 2n\dot{u} = \frac{\partial T}{\partial v}$$
$$\ddot{w} + n^2w = \frac{\partial T}{\partial w}$$

- For this we turn back to the Hill equations
- For L1 we have that only the u equation counts, v and w are zero, and there are no perturbations (T=0). The u equation becomes:

$$\ddot{u} = \frac{G.m_q}{R_h^2} = 3n^2u = 3\frac{G.(m_p + m_q)}{a^3}R_h$$

$$R_h^3 = \frac{m_q}{3(m_p + m_q)}a^3 \implies R_h = a.\left[\frac{m_q}{3(m_p + m_q)}\right]^{1/3}$$

Example problem 2

- Show that our Moon is within the Hill sphere of the Earth
- Currently the Moon disappears at a rate of 3.8 cm per year due to tidal energy dissipation.
- How long would it take before we lose our Moon because it leaves our Hill sphere?
- All constants can be found in the PS book

Planetary Perturbations and Resonances

Within the solar system all planets and Moons attract one another. This causes the Keplerian elements to change over time, susceptible elements are for instance the semi-major axis, the inclination and the eccentricity of asteroids and comets. Planetary orbit perturbations may be approximated by a second order system that is perturbed by an oscillation (see Hill equations):

$$\ddot{w} + n^2 w = A\cos\omega t + B\sin\omega t$$

The solution of this system is that of a harmonic oscillator:

$$w(t) = C_0 \cos nt + C_1 \sin nt + \frac{A\cos\omega t + B\sin\omega t}{n^2 - \omega^2}$$

This solution will resonate when n and ω approach one another

Examples of resonances in the solar system

- Several resonances locks such as Epimetheus-Janus (see Lagrange point discussion).

- But also Neptune-Pluto, etc etc

- Resonances may force material into highly eccentric orbits, this results in collisions which clears the Kirkwood gaps in the asteroid belt.
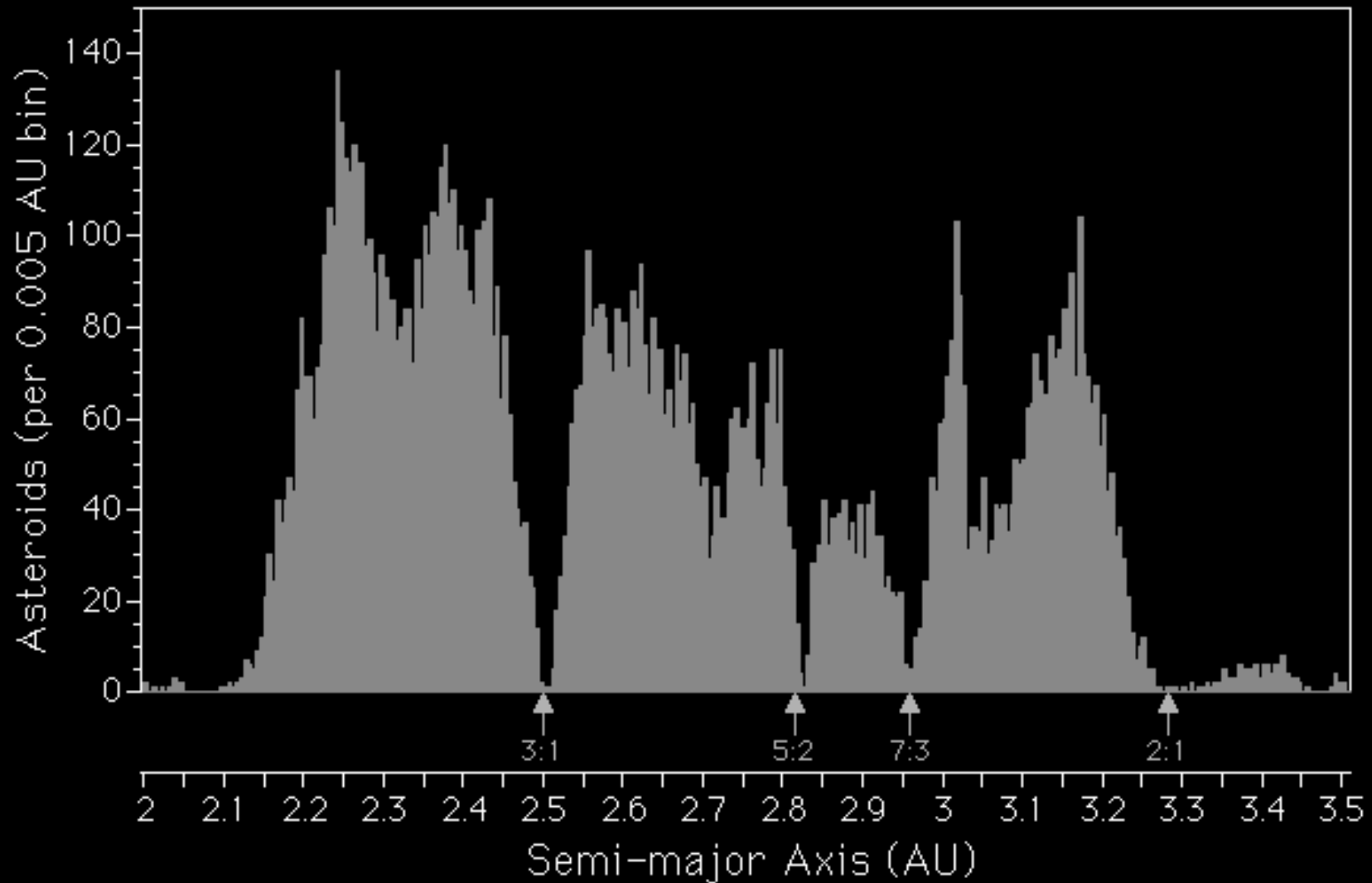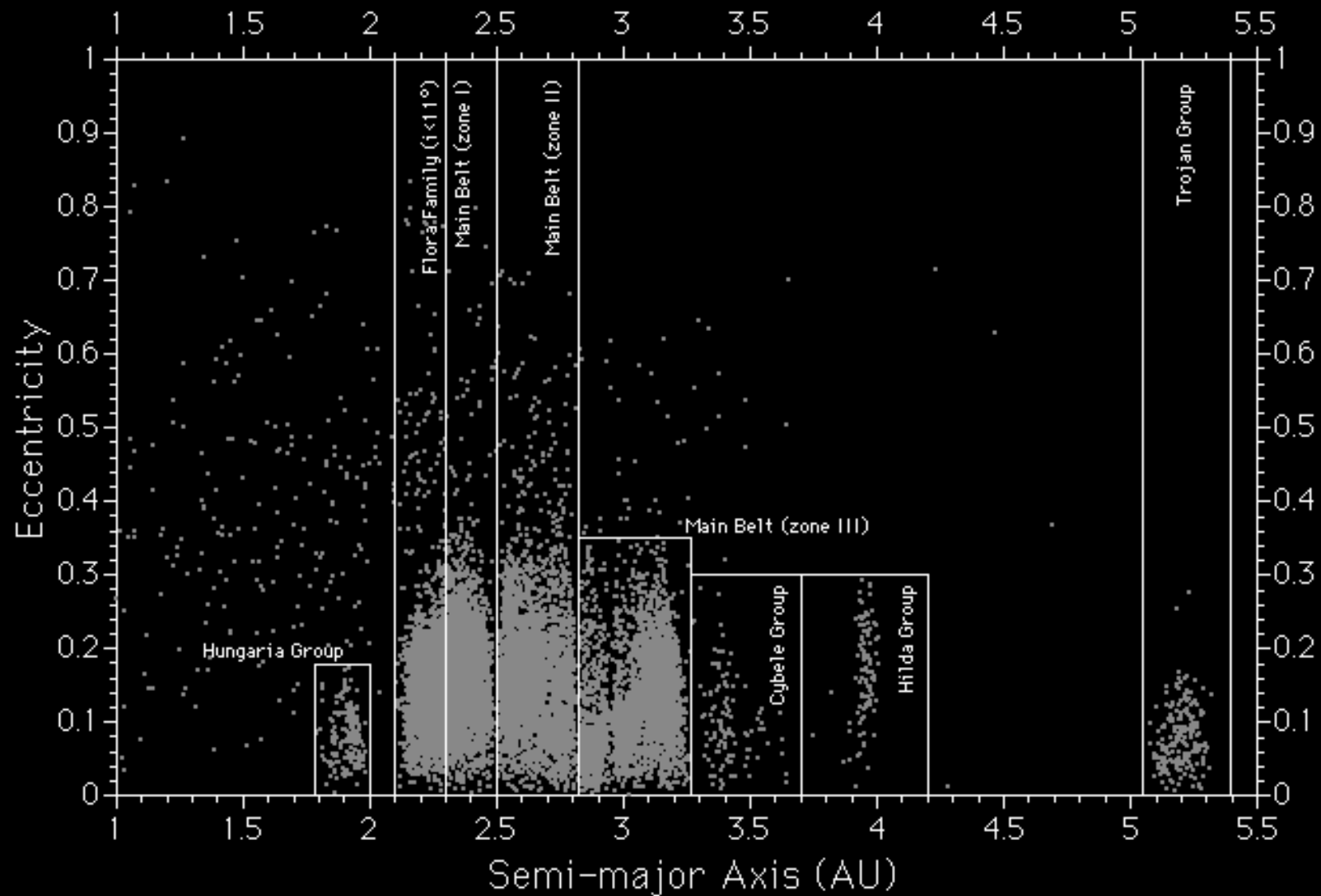
Compare to fig 9.1 in Planetary Sciences book   (JPL)

Asteroid Distribution
Inner Solar System

Compare to fig 9.1 in Planetary Sciences book   (JPL)

Long term stability of planetary orbits

- Why is the solar system as it currently is? The system is 4.5 billion years old.
- Are there periods of chaotic behavior in the solar system?
- Analytically you get resonances, but they are not real (moreover it is very difficult to do it right, see (Taff,1981))
- Numerical integration shows that the system remains stable for a very long time.
- The Earth's orbital elements become harmonic expressions, and there are 100k year variations in the orbital elements (Ice age cycles)

Orbits about an oblate planet

- Kepler's theory is good approximation of reality, and it describes the motion in gravity fields near large bodies like Earth or Sun pretty well.

- The potential of the field is what we called U for the Kepler problem, but, there are higher order moments which we ignored

- The complete gravitational potential V is formally expressed as V=U+T, where T is a distrubing potential.

- Gravitational flattening (a part of T) results in a special situation which does significantly affect orbits, and the result can be explained analytically

- In the following many mathematical details are skipped, we only show the solution! (otherwise you may want to attend the lectures on astrodynamics I and II).

Perturbed equations of motion

$$\ddot{\overline{X}} = -\nabla V + \sum_i \overline{F}^i$$

$$V = -\frac{GM}{r} - \frac{GM}{a_e} \sum_{n,m,a} \left(\frac{a_e}{r}\right)^{n+1} \overline{C}_{nma} \overline{Y}_{nma}(\varphi, \lambda)$$

$$\overline{Y}_{nma}(\varphi, \lambda) = \overline{P}_{nm}(\cos\theta) \begin{cases} \cos m\lambda & : & a = 0 \\ \sin m\lambda & : & a = 1 \end{cases}$$

Typical questions are: 1) what coordinate system do we use? 2) explain all symbols 3) what are the ranges of the indices n, m and a 4) calculate the gradient of V

Solution equations of motion

Analytic

- – Lagrange planetary equations
- – Gravity potential in Keplerian elements
- – Isolate the first-order solution
- – Approximate higher-order perturbations

Numeric

- – Conversion to system of first-order ODE
- – Integration of system of equations
- – (attend other lectures if you want to know more about the details)

Linear perturbations by C_{20}

$$\frac{d\Omega}{dt} = C_{20} \frac{3na_e^2}{2a^2(1-e^2)^2} \cos I$$

$$\frac{d\omega}{dt} = C_{20} \frac{3na_e^2}{4a^2(1-e^2)^2}(1-5\cos^2 I)$$

$$\frac{dM}{dt} = n - C_{20} \frac{3na_e^2}{4a^2(1-e^2)^{3/2}}(3\cos^2 I - 1)$$

Notes:

- C_{20} not normalised, n: mean motion

- Similar expressions for higher-order moments (see book)

Example problem 3

- For what inclination do we get a sun-synchronous orbit at 1000 km above the Earth's surface?

- For what inclination will the peri-apsis become fixed with respect to the planet?

Tides

- Newton once said:
    - A tidal force is caused by the gravitational attraction of a remote body on your planet.
    - The DIFFERENCE between the local force and the force acting on the center of gravity of your planets is relevant for tides
- On Earth tides are caused by Sun and Moon
- In reality everything else in the solar system matters in the exact formulation of tidal accelerations.

Tide generating force

Apollo 8 photograph of the earth rising above the lunar horizon
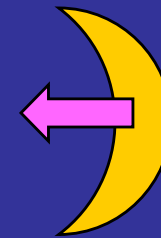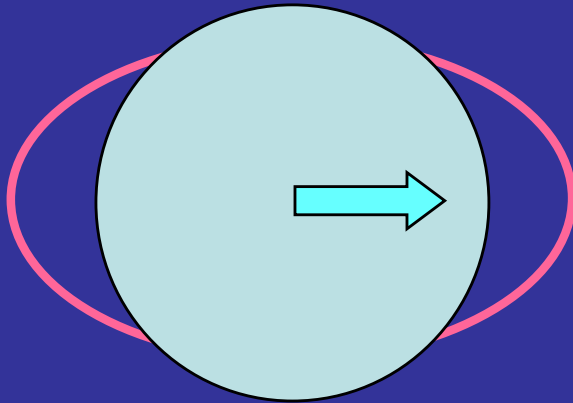
Roche limit

- Near a large planet, tidal forces grow beyond the maximum stress limit of a body causing it to break up, this limit is called the Roche's limit.

- The Roche limit is found by balancing the Moon's self gravity at the surface against the tidal force of a planet

Spokes in Saturn's B-Ring   © Copyright 2002 Calvin J. Hamilton
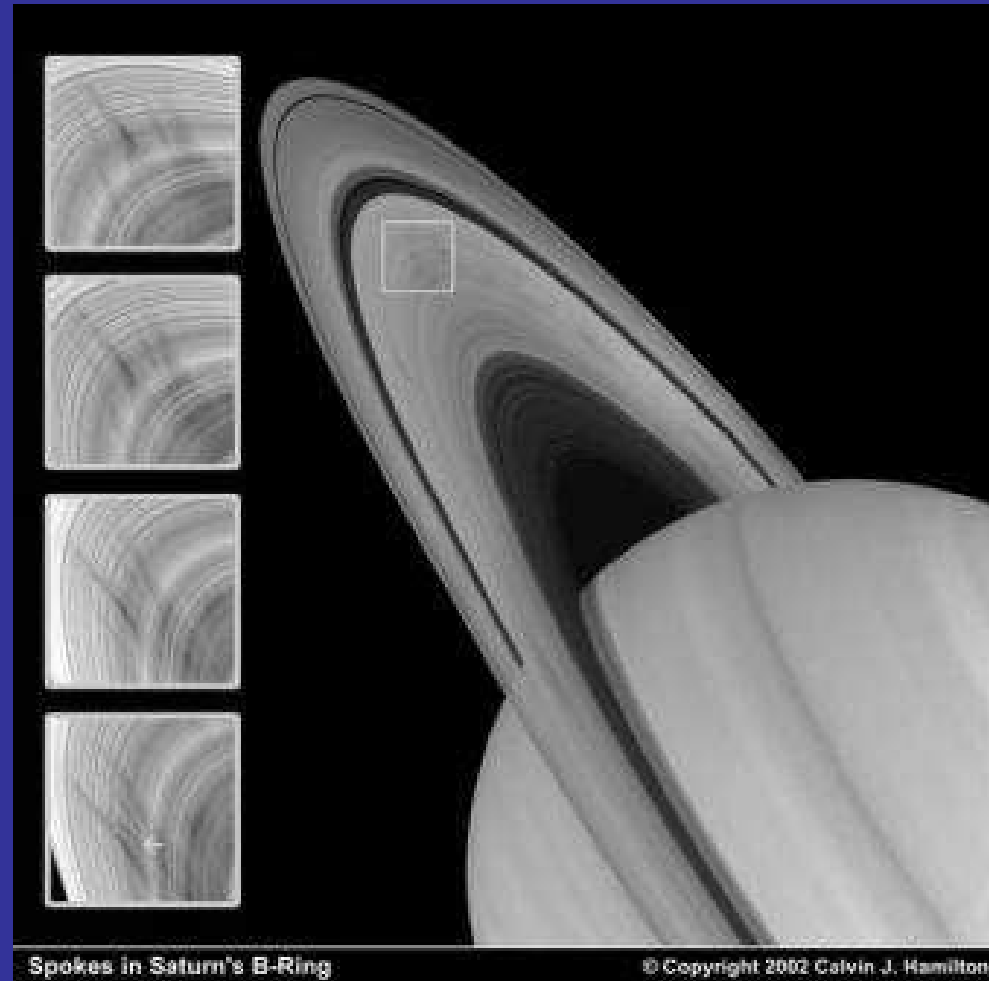
How large is the Roche limit

$$a_{tides} = a_{binding}$$
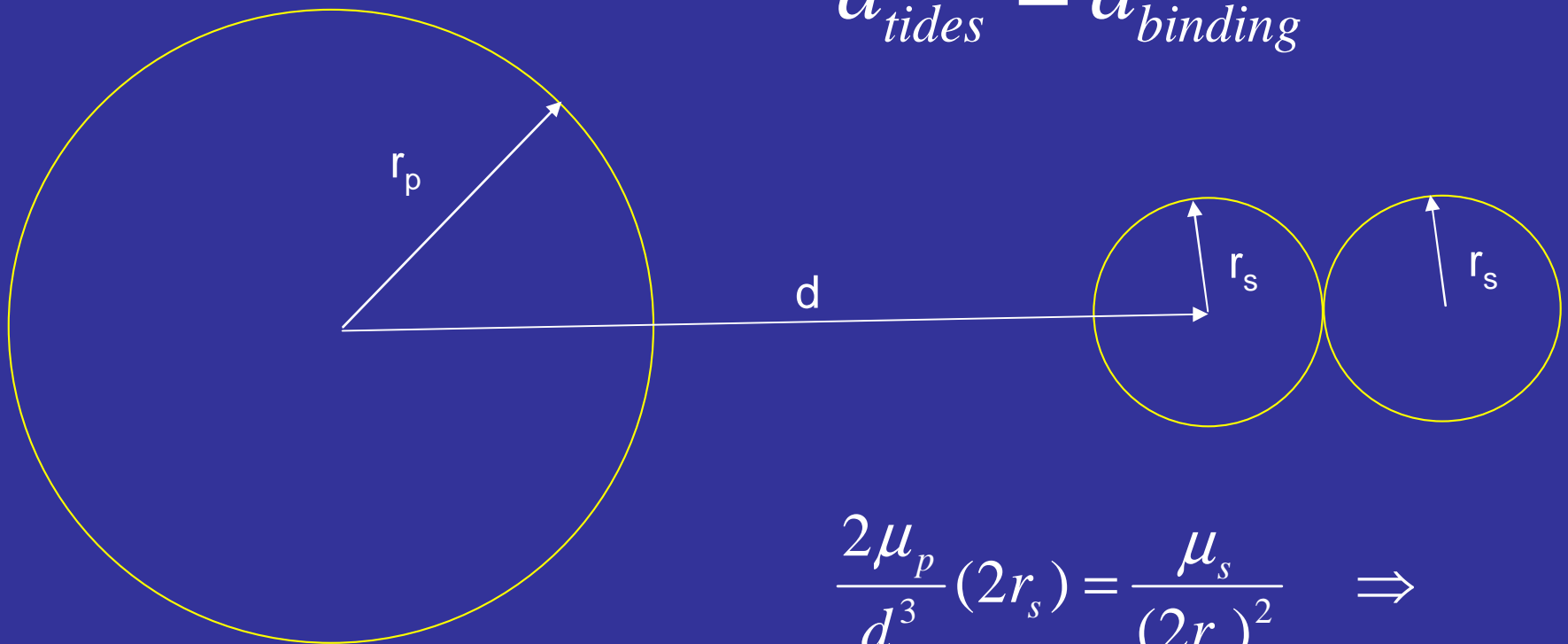
$$\frac{2\mu_p}{d^3}(2r_s) = \frac{\mu_s}{(2r_s)^2} \quad \Rightarrow$$

$$d \approx 2.52 \times r_p \times \left(\frac{\rho_p}{\rho_s}\right)^{1/3}$$

Some remarks

- Most Moons around planets are outside the Roche limit and within the Hill sphere

- The tensile strength of the Moon is not taken into account, but it does play a role

- There are Moons, like Amalthea around Jupiter and Pan around Saturn orbit within the Roche Limit.
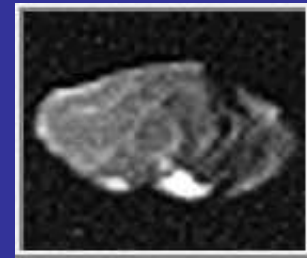
Amalthea
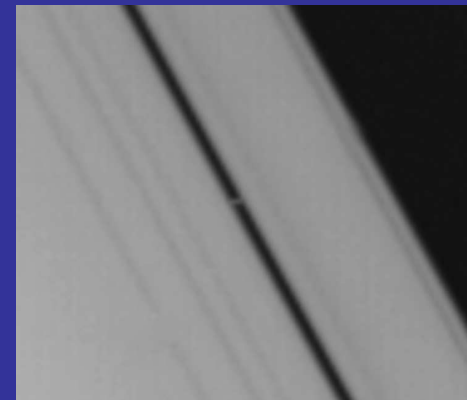
Pan

Tidal energy dissipation

3.82 cm/yr

M_2 : 2.50 +/- 0.05 TW
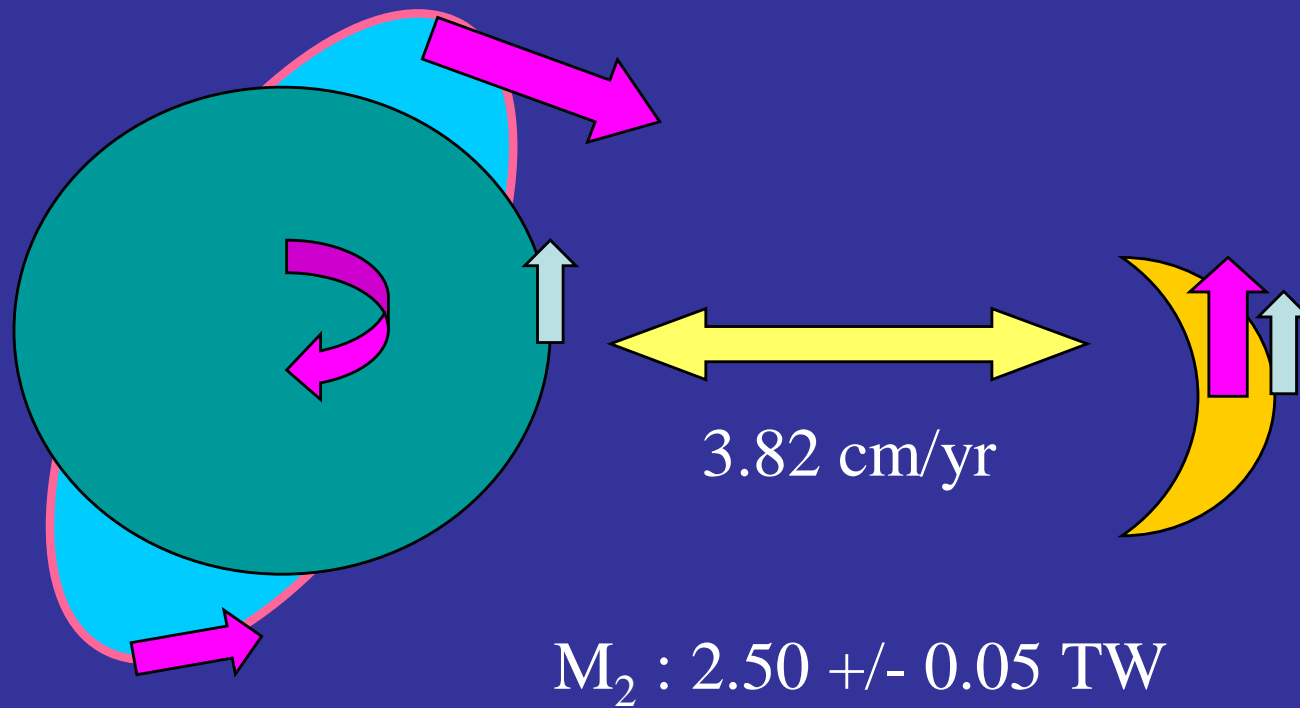
(Munk,1997)

M2 ocean tide
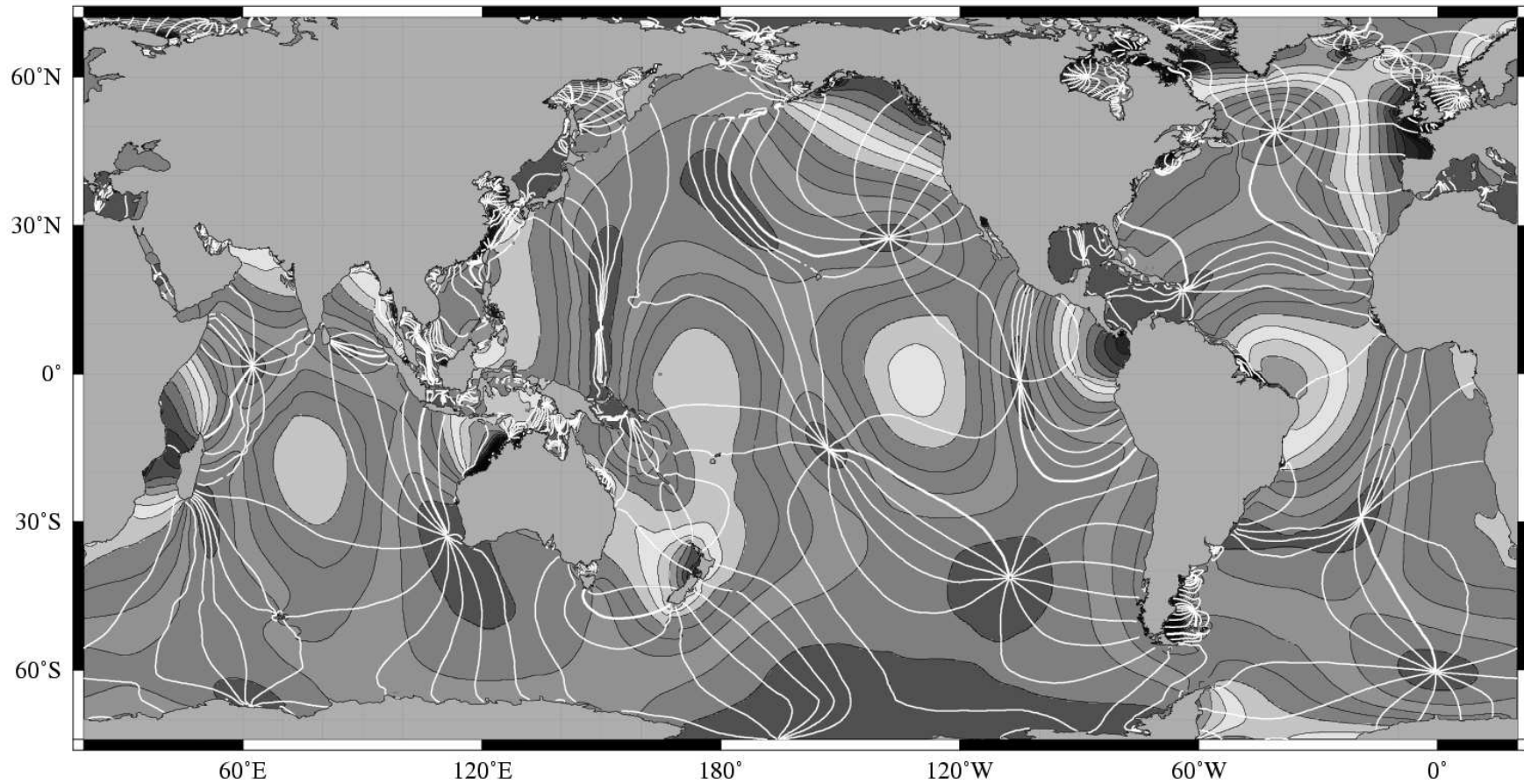
R Ray
Space Geodesy Branch

6/99

0 10 20 30 40 50 60 70 80 90 100 110 120 130 cm
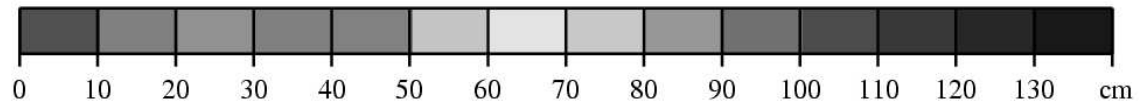
Tidal energy dissipation map
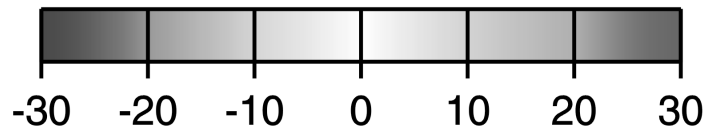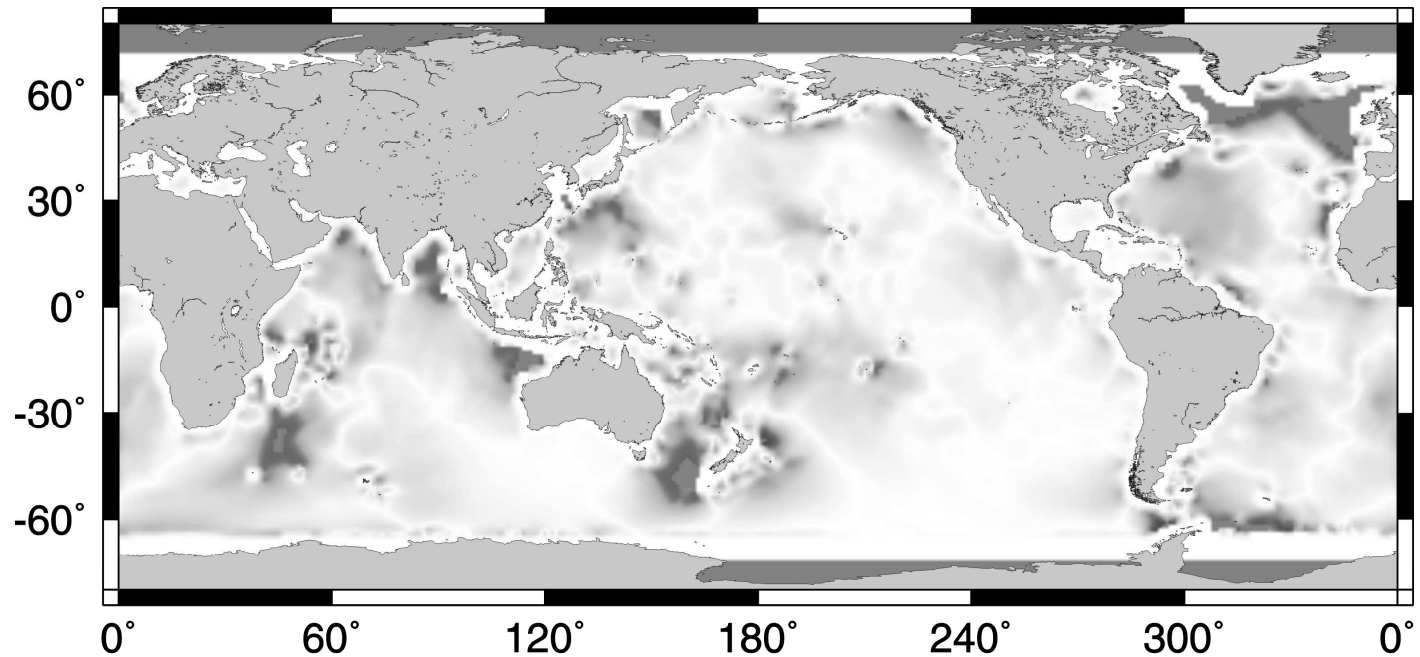
MilliWatt/m^2

Dissipative forces and the orbits of small satellites

- ## Radiation pressure
  - Sun light pushes particles (mostly micrometer sized dust) outwards
- ## Poynting-Roberton drag
  - Centimeter sized particles spiral inwards to the Sun
- ## Yarkovsky effect
  - Changes orbits of meter to km sized objects due to an uneven temperature distribution at the surface
- ## Corpusclar Drag
  - Like Poynting-Robertson, except that it is caused by solar wind particles
- ## Gas Drag
  - Gas (atmosphere or in interplanetary space) exerts a drag effect on a body proportional to the velocity squared of the body relative to the gas
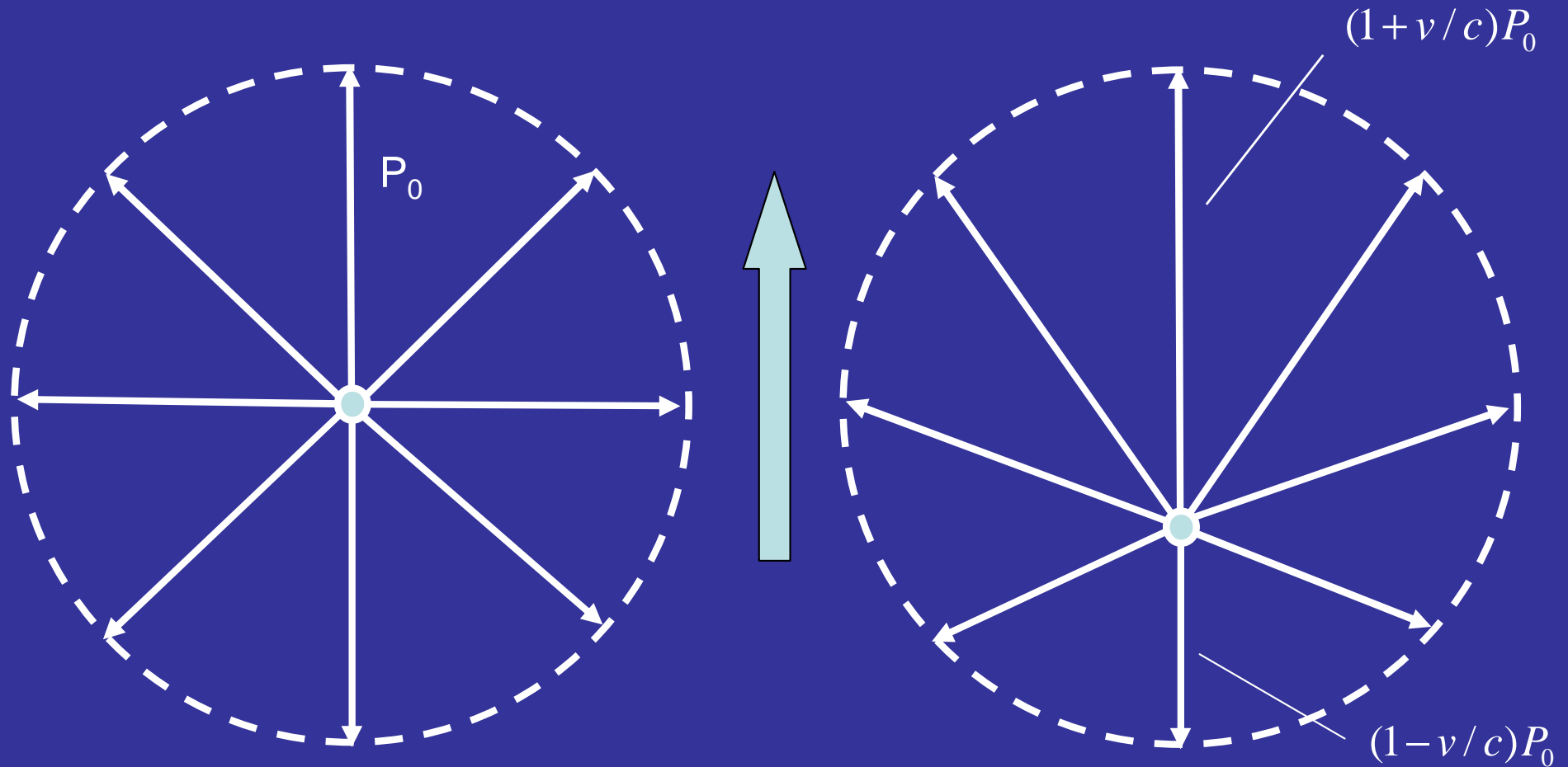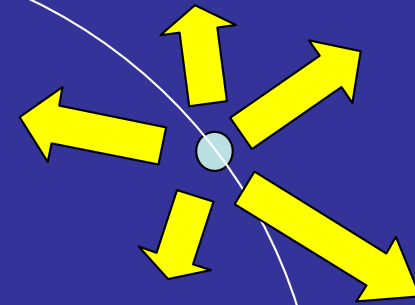
Radiation pressure
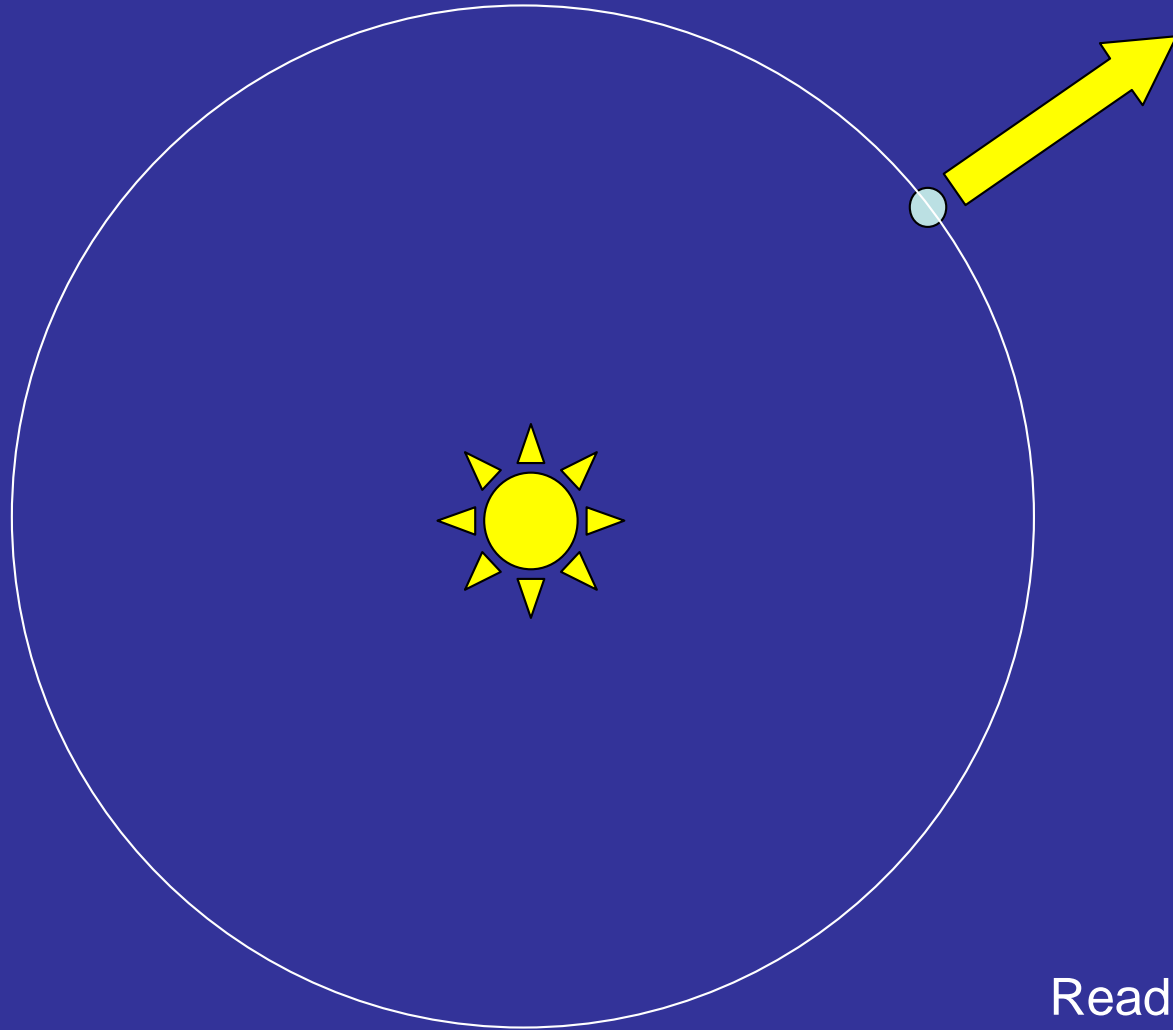
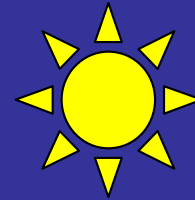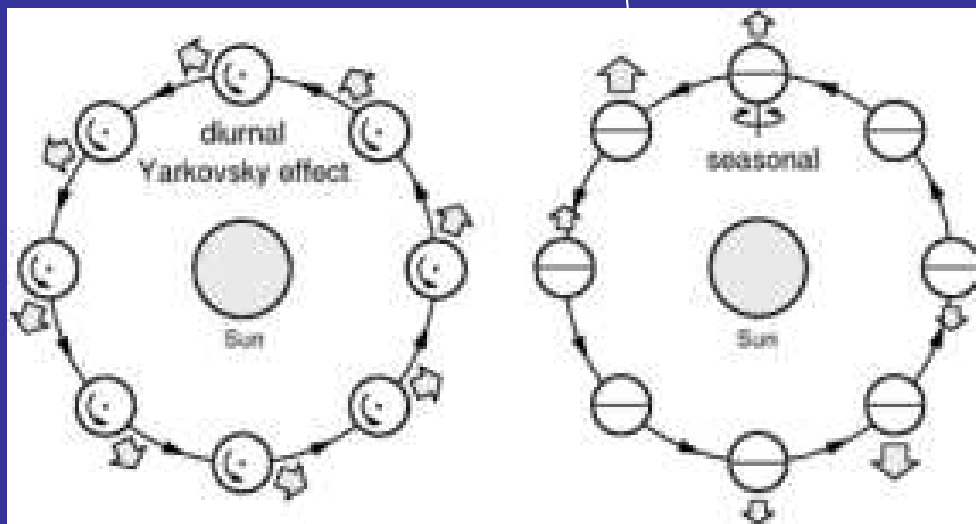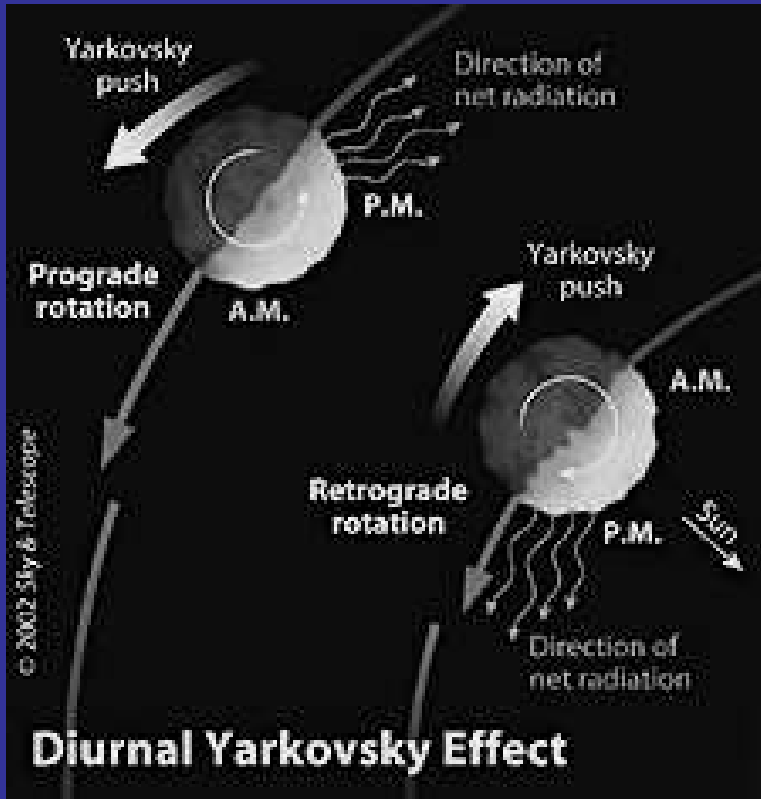Read: 2.7.1

Poynting Robertson Drag

For particles that re-radiate pressure the net effect is a drag in the flight direction. Analogy: If you bicycle in the rain, then you'll become wet even when there is no wind.
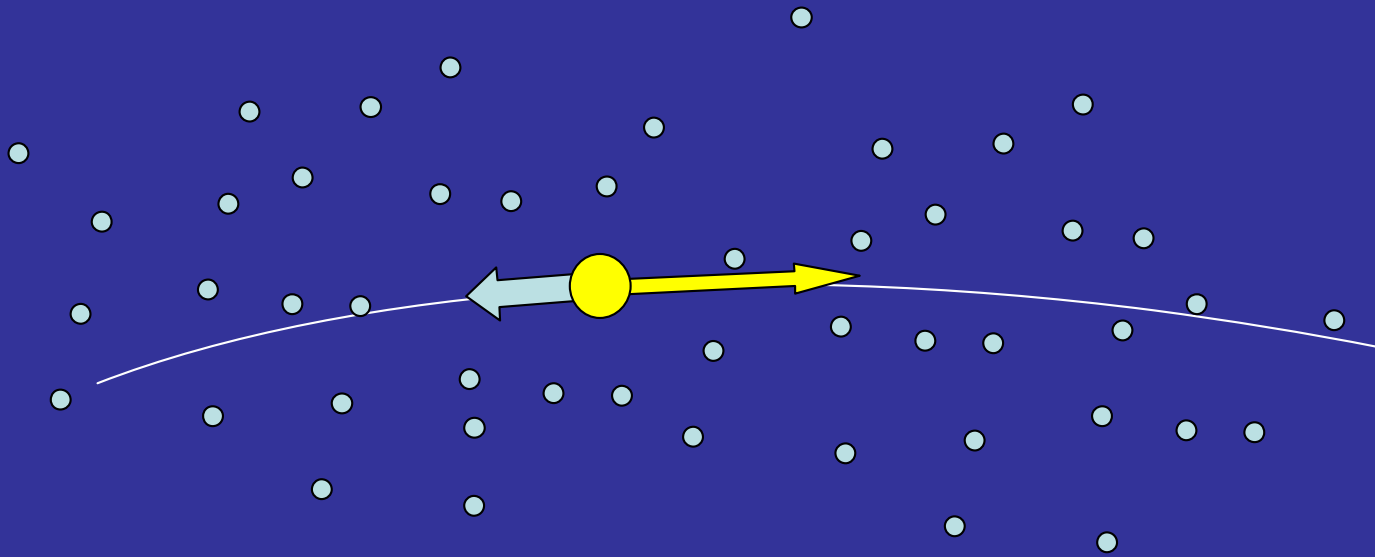
$(1+v/c)P_0$

P_0

$(1-v/c)P_0$

In Particle's Frame

In Rest Frame of Sun

Yarkovski Effect

Read: 2.7.3

Gas drag

$$F = \frac{1}{2} C_d \rho A v^2$$

Example problem 4

- Suppose that a satellite at 1000 km above the Earth loses 1 km per day due to air drag, the orbit remains circular, A = 10 square meter, C_d=3, mass = 1000 kg.

- What is the approximate density of the Earth's atmosphere in kg/m^3 for this configuration

Homework assignments

- For work lecture on 20-November:
    - Exercise 2.3 on page 39
    - Exercise 3.29 on page 64

Study/Read PS book

- 13 Nov (Solar system formation etc)
  - Study: 1.3, 12.1,
  - Read: Ch. 1, 12.3, 12.4.1, 12.6 – 12.9, 12.11, 12.12
- 20 Nov (Solar system dynamics)
  - Study: Ch. 2.1 , 2.2, 11.1,
  - Read: Ch. 2