Hill equations

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1 Hill vergelijkingen

G.W. Hill (1838-1914) considered equations of motions in a uniformly rotating coordinate system. Consider the inertial system $\overline{x}$ where the x and y axis appear within the orbital plane and where the z-axis is pointing in the direction of the angular momentum vector. Consider also the rotating $\overline{\alpha}$ system with the γ-axis coinciding with the z axis and where the α-axis is pointing at the satellite. The situation sketch is shown in figure 1. The

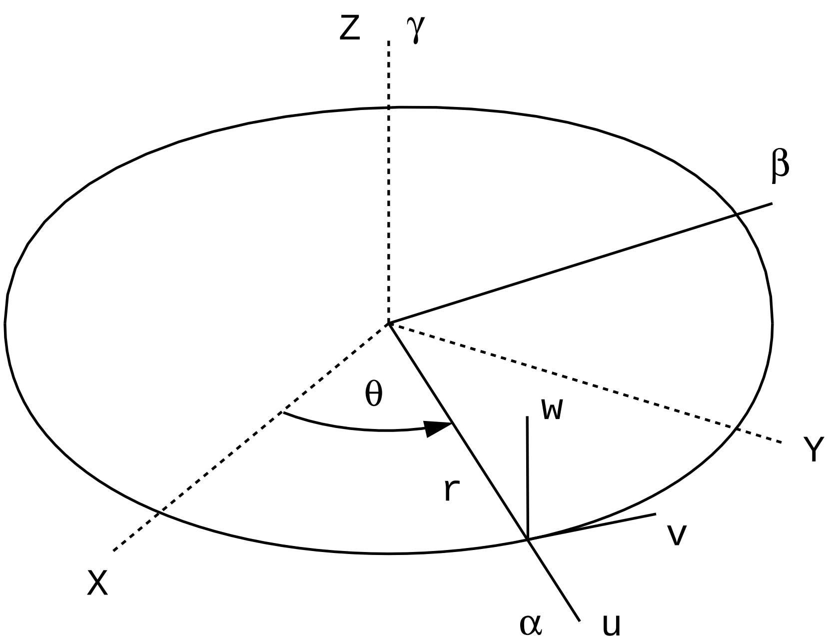

Figure 1: The $\overline{x}$ and the $\overline{\alpha}$ system.

relation between both systems is as follows:

$$\overline{x} = R(\theta)\overline{\alpha} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \tag{1}$$

and

$$\theta(t) = \theta_0 + \dot{\theta}t = \theta_0 + nt \tag{2}$$

where n is constant. The second order derivative of $\overline{x}$ with respect to time is:

$$\ddot{\overline{x}} = R\ddot{\overline{\alpha}} + 2\dot{R}\dot{\overline{\alpha}} + \ddot{R}\overline{\alpha} \tag{3}$$

so that:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = R(\theta) \begin{bmatrix} \ddot{\alpha} - 2n\dot{\beta} - n^2\alpha \\ \ddot{\beta} + 2n\dot{\alpha} - n^2\beta \\ \ddot{\gamma} \end{bmatrix} \tag{4}$$

The same is true for the gradient of the potential:

$$\begin{bmatrix} \partial V/\partial x \\ \partial V/\partial y \\ \partial V/\partial z \end{bmatrix} = R(\theta) \begin{bmatrix} \partial V/\partial \alpha \\ \partial V/\partial \beta \\ \partial V/\partial \gamma \end{bmatrix} \tag{5}$$

It is relatively easy to show that $R(\theta)$ contains the partial derivatives and that (5) follows from the chain rule. The equations of motion in the rotating $\overline{\alpha}$ system are obtained from eq. (4) and eq.(5):

$$\begin{aligned} \ddot{\alpha} - 2n\dot{\beta} - n^2\alpha &= \frac{\partial V}{\partial \alpha} \\ \ddot{\beta} + 2n\dot{\alpha} - n^2\beta &= \frac{\partial V}{\partial \beta} \\ \ddot{\gamma} &= \frac{\partial V}{\partial \gamma} \end{aligned} \tag{6}$$

The next step is to express the potential in the $\overline{\alpha}$ frame at the position of the satellite. This is accomplished by linearizing the expression $V = U + T$ with $U = \mu/r$ at the nominal orbit, cf. $\alpha = r$, $\beta = 0$ and $\gamma = 0$ where T is referred to as the disturbing potential. Furthermore u, v en w denote small displacements in the $\overline{\alpha}$ frame. ($u = \Delta\alpha$, $v = \Delta\beta$, $w = \Delta\gamma$.) The linearized gradient in the $\overline{\alpha}$ frame at the true position of the satellite (cf. $\alpha = r + u$, $\beta = v$ en $\gamma = w$) is:

$$\begin{aligned} \frac{\partial V}{\partial \alpha} &= -\frac{\mu}{r^2} + 2\frac{\mu}{r^3}u + \frac{\partial T}{\partial u} + \cdots \\ \frac{\partial V}{\partial \beta} &= -\frac{\mu}{r^3}v + \frac{\partial T}{\partial v} + \cdots \\ \frac{\partial V}{\partial \gamma} &= -\frac{\mu}{r^3}w + \frac{\partial T}{\partial w} + \cdots \end{aligned} \tag{7}$$

If we equate (6) to (7) then we find:

$$\begin{aligned} \ddot{u} - 2n\dot{v} - 3n^2 u &= \frac{\partial T}{\partial u} \\ \ddot{v} + 2n\dot{u} &= \frac{\partial T}{\partial v} \\ \ddot{w} + n^2 w &= \frac{\partial T}{\partial w} \end{aligned} \tag{8}$$

and these equations are known as the Hill equations.

2 Solution of the Hill equations

An interesting property of the Hill equations is that analytical solutions exist provided that n is constant. In this case eq. (8) becomes:

$$\dot{\overline{u}} = F\overline{u} + \overline{g} \tag{9}$$

where F does not depend on time and where $\overline{g}$ does depend on time. The homogeneous and the particular solutions are discussed in the following sections.

2.1 Homogeneous part

In this case $\overline{g} = 0$ and to obtain a solution we decompose F in eigenvalues Λ while the eigenvectors appear in the columns of the Q matrix:

$$F = Q\Lambda Q^t \tag{10}$$

The homogenous solution becomes:

$$\overline{u}(t) = Qe^{\Lambda.(t-t_0)}Q^t\overline{u}(t_0) \tag{11}$$

To demonstrate that this is a solution we consider a Taylor expansion of $u(t)$:

$$\overline{u}(t) = \overline{u}(t_0) + \dot{\overline{u}}(t_0)(t-t_0) + \frac{1}{2}\ddot{\overline{u}}(t_0)(t-t_0)^2 + ... + \frac{1}{n!}\overline{u}^{(n)}(t_0)(t-t_0)^n \tag{12}$$

with:

$$\begin{aligned} \dot{\overline{u}} &= F\overline{u} \\ \ddot{\overline{u}} &= \dot{F}\overline{u} + F\dot{\overline{u}} = F.F.\overline{u} = F^2\overline{u} \\ &\vdots \\ \frac{\partial^n \overline{u}}{dt^n} &= F^n\overline{u} \end{aligned}$$

so that:

$$\overline{u}(t) = (I + F.(t-t_0) + \frac{1}{2}F^2.(t-t_0)^2 + ... + \frac{1}{n!}F^n(t-t_0)^n + ...)\overline{u}(t_0)$$

from which it follows that:

$$\overline{u}(t) = e^{F.(t-t_0)}\overline{u}(t_0).$$

If $F = Q\Lambda Q^t$ then:

$$\overline{u}(t) = e^{Q\Lambda Q^t.(t-t_0)}\overline{u}(t_0),$$

and as a result:

$$e^{Q\Lambda Q^t.(t-t_0)} = Qe^{\Lambda.(t-t_0)}Q^t \tag{13}$$

This proves that eq.(11) is a solution of the homogeneous problem since Q is orthonormal. Since $\Lambda.(t-t_0)$ is diagonal $e^{\Lambda.(t-t_0)}$ is easily obtained, for real λ_i on the diagonal of Λ we obtain exponential terms that either decay for negative eigenvalues or that grow for positive eigenvalues and for complex eigenvalues λ_i we find oscillating solutions.

2.2 Particular solution

For the homogeneous part we found:

$$y' + a\,y = 0 \tag{14}$$

where

$$\Phi(x) = c\mathrm{e}^{-ax}$$

is a solution. The proof follows after a substitution of:

$$\Phi'(x) = -c\,a\mathrm{e}^{-ax} \tag{15}$$

in eq. (14). For the particular solution we need:

$$y' + ay = b(x) \tag{16}$$

for which it can be shown that:

$$\Phi(x) = c\mathrm{e}^{-ax} + \mathrm{e}^{-ax}B(x) \tag{17}$$

with

$$B(x) = \int_{x_0}^{x} \mathrm{e}^{at}b(t)\,dt \tag{18}$$

is a solution. This can be shown by substitution of $\Phi'(x)$ and $B'(x)$ in eq. (16). We remind that Eq. (18) is known as the Laplace transform of $b(x)$ and that Laplace transforms of most functions are known. To demonstrate that the same technique can be used for a system of differential equations we consider the system:

$$\dot{\overline{y}} + A\overline{y} = \overline{b}(x) \tag{19}$$

where we decompose A as:

$$A = Q\Lambda Q^T$$

and where we pre-multiply with Q^t:

$$Q^T\dot{\overline{y}} + \Lambda Q^T\overline{y} = Q^T\overline{b}(x)$$

By equating $\overline{z} = Q^T\overline{y}$ we obtain:

$$\dot{\overline{z}} + \Lambda\overline{z} = \overline{c}(x)$$

so that we obtain a decoupled system. In this case the solution is:

$$\overline{\Phi}(x) = d\mathrm{e}^{-\Lambda x} + \mathrm{e}^{-\Lambda x}\overline{C}(x) \tag{20}$$

$$\overline{C}(x) = \int_{x_0}^{x} \mathrm{e}^{\Lambda t}\overline{c}(t)\, dt \tag{21}$$

where

$$\overline{y}(t) = Q\ \overline{\Phi}(x) \tag{22}$$

is a solution of (19).

3 Problems

- Find the homogeneous solution of the Hill equations, how do you interpret this homogeneous solution.
- Find the particular solution of the Hill equations where the forcing function is expressed as $A\cos\omega t + B\sin\omega t$. Show that the particular solutions result in a linear response of u v and w relative to the forcing terms.