# Lagrangian Points

E. Schrama, e-mail: e.j.o.schrama@tudelft.nl

April 12, 2006

## 1 Introduction

The two-body problem in celestrial mechanics leads to the description of a small satellite moving in an orbital plane where the motion is entirely forced by a planet whose mass is far greater than that of the satellite. The motions are usually circular or elliptical; the orbital period depends on the mass of the main planet and the semi-major axis of the satellite.

The natural question arrises what would happen if we added another planet to this problem whose mass is about the same order as that of the first planet. There are no easy solutions for the general three body problem, but there are solutions for the restricted three-body problem depicted in figure 1.

## 2 The restricted three-body problem

In figure 1 we assume that planet $P$ with mass $m_p$ is located at $(0, -d_p)$ and that planet $Q$ with mass $m_q$ is at $(0, +d_q)$ . This system is rotating with a constant angular speed $n$ about a center of mass which is the meeting point of all dashed lines in figure 1.

The angular rotation rate will now depend on the sum of the masses of the planets and sum of $d_p$ and $d_q$, these distances follow in turn from the masses $m_p$ and $m_q$. The center of mass point is often referred to as the barycenter of the system.

To maintain this configuration we have to demand that the sum of the centrifugal and the gravitational contributions of the acceleration balance one another for each planet. Therefore:

$$n^2 d_p = \frac{1}{m_p} \frac{G m_p m_q}{(d_p + d_q)^2} = \frac{\mu_q}{(d_p + d_q)^2} \tag{1}$$

$$n^2 d_q = \frac{1}{m_q} \frac{G m_p m_q}{(d_p + d_q)^2} = \frac{\mu_p}{(d_p + d_q)^2} \tag{2}$$

from which get:

$$n^2 = \frac{\mu_p + \mu_q}{(d_p + d_q)^3} \tag{3}$$

An essential step is to assume that $n$ is constant. We derive the equations of motion in this system by introducing a transformation for $\overline{\alpha}$ in the rotating system that relates to inertial positions in the inertial system $\overline{x}$:

$$\overline{x} = R_3(\theta)\overline{\alpha} \tag{4}$$

where $\theta(t) = n.(t - t_0)$ is a linear rotation angle and $R_3$ the rotation matrix. By straightforward differentiation we get:

$$\ddot{\overline{x}} = \begin{bmatrix} \ddot{\alpha}_1 - 2n\dot{\alpha}_2 - n^2\alpha_1 \\ \ddot{\alpha}_2 + 2n\dot{\alpha}_1 - n^2\alpha_2 \\ \ddot{\alpha}_3 \end{bmatrix} = \frac{-\mu_p}{|\overline{\alpha} - \overline{\alpha}_p|^3}(\overline{\alpha} - \overline{\alpha}_p) + \frac{-\mu_q}{|\overline{\alpha} - \overline{\alpha}_q|^3}(\overline{\alpha} - \overline{\alpha}_q) \qquad (5)$$

where $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is the position of the satellite and where $\overline{\alpha}_p$ and $\overline{\alpha}_q$ are the positions of $P$ and $Q$ in the $\alpha$ frame. If we exclude a relative velocity $\dot{\overline{\alpha}}$ (which would introduce a Coriolis effect) and if we constrain the motion to a plane ($x_3 = \alpha_3 = 0$) then we obtain the accelerations $\ddot{\overline{\alpha}}$ experienced by a co-rotating satellite. This satellite feels a combination of centrifigal and gravitational accelerations, and it leads to:

$$\ddot{\alpha}_1 = -\mu_p \frac{\alpha_1 + d_p}{|\overline{\alpha} - \overline{\alpha}_p|^3} - \mu_q \frac{\alpha_1 - d_q}{|\overline{\alpha} - \overline{\alpha}_q|^3} - n^2\alpha_1 \qquad (6)$$

$$\ddot{\alpha}_2 = -\mu_p \frac{\alpha_2}{|\overline{\alpha} - \overline{\alpha}_p|^3} - \mu_q \frac{\alpha_2}{|\overline{\alpha} - \overline{\alpha}_q|^3} - n^2\alpha_2 \qquad (7)$$

The length of the acceleration vector $|\ddot{\overline{\alpha}}|$ can now be plotted as a function of the position in the $\alpha$ frame. This is done in figure 2 where we have assumed a configuration with $d_p = 1$, $d_q = 10$, $\mu_p = 10$ and $\mu_q = 1$.

The Lagrangian points appear in "gravity-wells" which are located in the blue regions. The first "well" is the C shaped "horse-shoe" where $L_3$, $L_4$ and $L_5$ can be found in the white exclusion zones. The second "well" is between $P$ and $Q$, the third well is located behind $Q$ when facing it from $P$. Satellites can be captured within these gravity wells as long as their own residual velocity is low enough not adding too much "energy" for an escape from the well. Examples of this phenomenon are shown in [1].

In this figure we have ignored large accelerations which are in the neighborhood of $P$ and $Q$ where the local gravitational effect is dominating. Furthermore we have ignored to plot $|\ddot{\overline{\alpha}}|$ in the outer region where the rotational accelerations are dominating. In figure 1 we indicate the corresponding Lagrangian points $L_1$ to $L_5$ where a satellite would not experience any residual acceleration because $|\ddot{\overline{\alpha}}| = 0$. The reason is that there is a balance between gravitational and centrifugal accelerations.

# 3 Positions of Lagrangian points

Lagrangian points $L_1$ $L_2$ and $L_3$ appear on the line connecting the planets, and $L_4$ and $L_5$ appear at angles $\pm 60°$ relative to this line. To compute the positions of $L_1$, $L_2$ and $L_3$ we assume that $\alpha_2 = 0$, which directly results in $\ddot{\alpha}_2 = 0$ which is necessary for finding these points. The condition that $\ddot{\alpha}_1 = 0$ follows from the condition:

$$s - \frac{\mu_p(s + \mu_q)}{|s + \mu_q|^3} - \frac{\mu_q(s - \mu_p)}{|s - \mu_p|^3} = 0 \qquad (8)$$

where $s$ is the ordinate along the connection line counted from the center of mass of the system. Examples of $s$ for typical values of $\mu_p$ and $\mu_q$ (whose sum has to be 1 for this problem) are shown in table 1. Lagrangian points $L_4$ and $L_5$ are easy to find. In this case we assume that $|\overline{\alpha}| \approx 1$ and $|\overline{\alpha} - \overline{\alpha}_p| \approx 1$ and $|\overline{\alpha} - \overline{\alpha}_q| \approx 1$ which is valid when $\mu_p >> \mu_q$ so that $\ddot{\overline{\alpha}} = \overline{0}$ which directly follows from eq. (5). This situation is only possible for the configuration where the Lagrangian points are located on the top of a triangle with sides of length 1, see also figures 1 and 2. As a result $L_4$ and $L_5$ are located on a line to the center of mass that is at an angle $\pm 60°$ relative to the line connecting both planets.

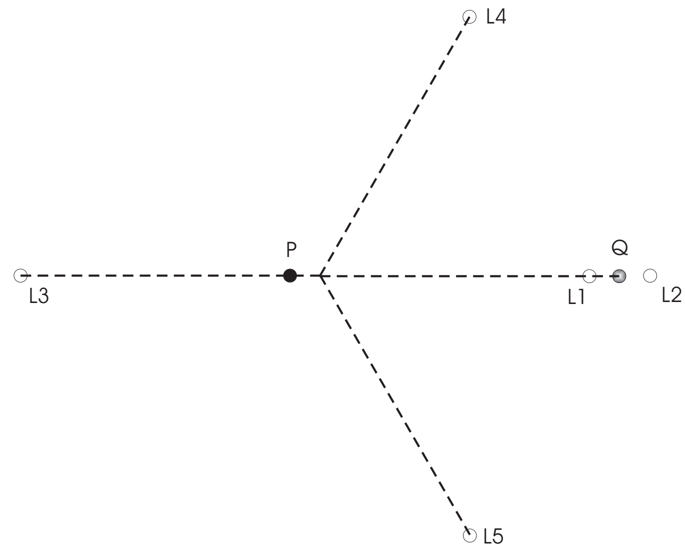


Figure 1: The restricted three body problem, with the positions of Lagrangian points are indicated by open circles

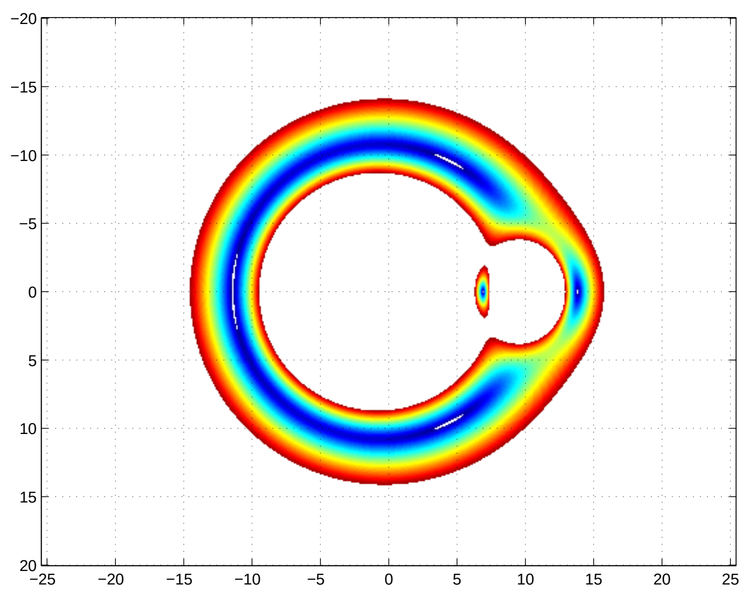


Figure 2: Colors indicate the length of the local acceleration vector in the rotating coordinate system (blue colors indicate shorter values than red ones). This configuration is computed for $\mu_p = 10$, $\mu_q = 1$ and correspondingly $\overline{\alpha}_p = (-1, 0)$ and $\overline{\alpha}_q = (0, 10)$.

| $\mu_p$ | $\mu_q$ | $s$ at $L_2$ | $s$ at $L_1$ | $s$ at $L_3$ |
|---|---|---|---|---|
| 0.9 | 0.1 | 1.259699833 | 0.6090351100 | -1.041608909 |
| 0.99 | 0.01 | 1.146765042 | 0.8480787130 | -1.004166612 |
| 0.999 | 0.001 | 1.069916098 | 0.9312869755 | -1.000416667 |

Table 1: Positions of Lagrange points $L_1$ to $L_3$

# References


[1] de Pater and Lissauer, Planetary Sciences, Cambridge University Press, Cambridge UK, 2004.